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The unification of hypo-plastic and elasto-plastic theories

Itai Einav *

Particles and Grains Laboratory, School of Civil Engineering, The University of Sydney, Sydney, NSW 2006, Australia

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ABSTRACT

A unifying theory of plasticity is developed that allows deducing models with either hypo-plastic, elasto-plastic or hyper-plastic constitutive structures. Elasto-plasticity is shown to be a singular case of hypo-plasticity. It is further demonstrated that certain conditions can be imposed to generate a new hierarchy of thermodynamically consistent hypo-plastic models, with a hyper-plastic structure as a singular case. The unifying theory is powerfully bridging between the tools created specifically for advancing models under either of those previous formulations.

1. Introduction

Elasto-plasticity and hypo-plasticity are two successful and popular frameworks of plasticity, at least in geomechanics. Elasto-plasticity describes models that distinguish between purely elastic and inelastic constitutive responses, through a notional yield surface (Hill, 1950; Drucker et al., 1957). This graphical concept is attractive for pedagogical purposes, but ignores the dissipation presented by many materials even prior to that yield, as evidenced by acoustic emission (e.g., from rocks, Baud et al., 2004; from granular materials, Fernandes et al., 2004) and infrared imaging of heat production (e.g., from metal materials, Benallal et al., 2008). Under certain conditions elasto-plastic models can be devised in such a way that satisfies the laws of thermodynamics (e.g., Ziegler, 1983; Chandler, 1985; Collins and Houlsby, 1997). Initially, the term 'hyper-plasticity' was coined for elasto-plastic models derivable from potentials, or pseudo-potentials (Wu and Kolymbas, 1990, W&K). Later, Houlsby and Puzrin (2006, H&P) adapted the use of this term for elasto-plastic models that satisfy the laws of thermodynamics, and in addition required a group of procedures for obtaining flow rules in terms of the dissipation function. H&P's definition of 'hyper-plasticity' is more restricting compared to that by W&K, but it does not allow deriving all models of elasto-plasticity consistent with thermodynamics. Therefore, in the current paper the term 'hyper-plasticity' is used to reflect all elasto-plastic models that satisfy the laws of thermodynamics.

In contrast, at their historical outset, models of hypo-plasticity have been openly devised without thermodynamic considerations, energy potentials, or any form of potential function whose partial derivatives determine the constitutive rate equation (Wu and Kolymbas, 1990; Kolymbas, 1991; Tamagnini et al., 2000). This freedom from thermodynamics allowed the models of hypo-plasticity to embrace a more predictive incrementally non-linear format (e.g. Chambon and Renoud-Lias, 1979; Darve and Labanieh, 1982) that impressively captures the material's stress–strain phenomenology during complicated cyclic loadings (Niemunis, 1993).

Recent work in granular physics has led to the definition of a limiting number of hypo-plastic models which, in fact, satisfy the laws of thermodynamics (Svendsen et al., 1999; Jiang and Liu, 2007, 2009; Gudehus et al., 2011). In particular, Jiang and Liu make a milestone step towards a comprehensive granular hydrodynamic model for dry granular systems, such as sand, which reveals a hypo-plastic structure. However, it is hard to conceive how their model could apply to other soils, such as clays, or in fact to any other solid material. On the other hand, the inherent representation of yield in elasto-plasticity and hyper-plasticity has been proven to be a good starting point for capturing inelastic deformations of many solid materials, particularly as they accumulate towards a well prescribed failure. Failure representation is a deficiency of most current hypo-plastic models, where particular load-paths can be followed that violate the desired failure criterion. Two exceptional contributions in that respect, which did address this difficulty of hypo-plasticity, are due to Chambon (e.g., Chambon et al., 1994; Wu and Niemunis, 1996). Nevertheless, these solutions are limited to the models proposed. Compared with hypo-plasticity, elasto-plasticity has found many more applications by solid mechanicians (i.e., beyond soil mechanics); this may possibly be attributed to the disassociation of hypo-plasticity from thermodynamics. However, there should be no doubt that the phenomenology content of hypo-plasticity has a huge potential beyond soil mechanics which should not be overlooked by solid mechanicians, especially when

* Tel.: +61 2 935 12113; fax: +61 2 935 13343.
E-mail address: itai.einav@sydney.edu.au

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concerned with cyclic loading. In the following a unifying mathematical framework will be presented that is able to bridge between the two frameworks of hypo and hyper plasticity; in achieving this, the difficulties above will then be resolved.

For example, in unifying these two frameworks the mathematical tools established for hyper-plasticity will help removing the difficulty of hypo-plasticity to fulfil failure criteria, without neglecting the first and second laws of thermodynamics. In doing that, a continuous mathematical manifold will be mapped between those extremes; here, it will be shown that elasto-plasticity is simply a singular case of hypo-plasticity. The range of *hypo*-plastic models that satisfy the thermodynamic laws is extended with *hyper*-plasticity as a singular case. Adopting the previous definitions, those models are not hypo-plastic in the sense that they are derivable from potentials, consistent with thermodynamics; they are also not hyper-plastic since they are capable to involve dissipation continuously without artificially splitting between regimes of elasticity and elasto-plasticity. In the spirit of unification we term the new theory 'h^2plasticity'. Its hierarchical position compared to previous theories is shown in Fig. 1. It is important to note some possible links to other well-established 'incrementally non-linear' frameworks of plasticity, specifically to generalized plasticity (Zienkiewicz and Mroz, 1984; Pastor et al., 1990) and bounding surface plasticity (Dafalias and Popov, 1975; Bardet, 1990), which have all been formulated without any thermodynamic considerations.

The formulation developed in this paper will be constructed under the assumption of isothermal conditions and small deformations, with elastic strain being the only internal variable; future works will include multiple internal variables and address large deformations and thermal effects. The paper starts by a short presentation of hypo-plasticity Section 2. Section 3 presents the general conditions all models of plasticity (hypo and elasto) must satisfy according to thermodynamics (under the assumption of isothermal small strains; temperature dependence and finite deformations are topics for future extensions). The thermodynamics conditions discussed in Section 3 are then utilised in Section 4 for generating elasto-plastic (i.e., hyper-plastic) models; the same conditions are then used in Section 5 to construct the more general h^2 plasticity theory. Next, a one-dimensional motivating example is presented in Section 6 within the context of the new theory that reveals full transition from hypo-plasticity to elasto-plasticity (or hyper-plasticity). The remaining sections demonstrate the wide applicability of the new formulation for constructing multi-dimensional models, while addressing issues of failure and non-associativity.

2. Hypo-plasticity

There have been many proposed forms of hypo-plastic constitutive stress–strain equations. A rather general class of hypo-plastic models could be specified in the following way

$$\dot{\sigma}_{ij} = E_{ijkl}(\sigma_{ij})\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^r_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij})\right), \quad \text{with } \dot{\varepsilon}^r_{kl}(\sigma_{ij},\alpha\dot{\varepsilon}_{ij}) \neq \alpha\dot{\varepsilon}^r_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij}), \quad \forall\alpha<0 \tag{2.1a,b}$$

where σ_{ij} and $\dot{\sigma}_{ij}$ are the stress tensor and its rate; $\dot{\varepsilon}_{ij}$ and $\dot{\varepsilon}^r_{ij}$ are the total and relaxation strain rate tensors; and E_{ijkl} is a fourth-ranked tensor, later to be connected to the elasticity stiffness tensor in elasto-plasticity.

It follows that in hypo-plastic models $\dot{\sigma}_{ij}(\sigma_{ij},-\dot{\varepsilon}_{ij}) \neq -\dot{\sigma}_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij})$. In elasto-plastic models this relation does not hold 'thoroughly' since it is met only when the state of stress σ_{ij} is situated on the yield surface. It is this property that uniquely distinguishes hypo-plastic models from elasto-plastic, as they are basically 'incrementally thoroughly non-linear' (Chambon et al., 1994).

At the same time, rate-independent plastic models satisfy:

$$\dot{\varepsilon}^r_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij}dt) = \dot{\varepsilon}^r_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij})dt, \quad \forall dt>0 \tag{2.2}$$

from which it follows that $\dot{\sigma}_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij}dt) = \dot{\sigma}_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij})dt$, $\forall dt>0$. For example, a rather popular form of hypo-plastic models (Chambon et al., 1994) is often specified by taking $\dot{\varepsilon}^r_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij}) = f_{kl}(\sigma_{ij})\|\dot{\varepsilon}_{ij}\|$:

$$\dot{\sigma}_{ij} = E_{ijkl}(\sigma_{ij})\left(\dot{\varepsilon}_{kl} - f_{kl}(\sigma_{ij})\|\dot{\varepsilon}_{ij}\|\right) \tag{2.3}$$

where $\|\dot{\varepsilon}_{ij}\| = \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}$. Since $\|(\alpha\dot{\varepsilon}_{ij})\| \neq \alpha\|\dot{\varepsilon}_{ij}\|, \forall\alpha<0$, the model is incrementally thoroughly non-linear, but since $\|(\dot{\varepsilon}_{ij}dt)\| = dt\|\dot{\varepsilon}_{ij}\|$, $\forall dt>0$, the model is also rate independent. Until now, there have not been clear restrictions developed to ensure that such constitutive equations would not violate the laws of thermodynamics. This will be achieved in the current paper.

3. General thermodynamic requirements to formulating models of plasticity

A convenient way to express the energy conservation restriction of thermodynamics to modelling the isothermal constitutive stress–strain response of materials is given by:

$$\widetilde{W} = \dot{\Psi} + \widetilde{\Phi} \tag{3.1}$$

where $\widetilde{W} = \sigma_{ij}\dot{\varepsilon}_{ij}$ is the mechanical rate of work; Ψ and $\dot{\Psi}$ are the Helmholtz free energy potential and its rate of change, and $\widetilde{\Phi}$ is the rate of mechanical dissipation. The overhead tilde sign '$\sim$' represents a pseudo time-derivative. In using 'pseudo' it is highlighted that only the rates of (mechanical) dissipation and work, $\widetilde{\Phi}$ and $\widetilde{W}$, can be defined. The cumulative value of the rate of dissipation (dissipation) is therefore not an appropriate potential, but the rate of dissipation may be considered as a pseudo-potential for derivation of dissipative flow rates (Ziegler, 1983). Moreover, from the second law of thermodynamics the rate of dissipation must be non-negative,

$$\widetilde{\Phi} \geqslant 0. \tag{3.2}$$

Furthermore, let us consider a particular form of the Helmholtz free energy potential of plasticity models, only in terms of the elastic strains,

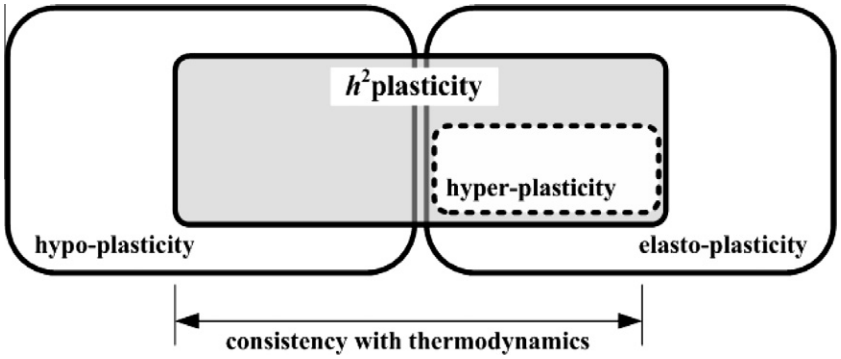

Fig. 1. Hierarchical place of the new formulation next to other, well-known theories of plasticity.

$$\Psi = \Psi\left(\varepsilon^e_{ij}\right). \tag{3.3}$$

Notice that we avoid employing the more general classification $\Psi = \Psi(\varepsilon^e_{ij}, \varepsilon^p_{ij})$. There are certainly no mathematical difficulties to proceed with this more general class of models, but such a step presents physical inconsistencies we wish to avoid. In particular, the dependence of the energy on the cumulative plastic strain conflicts with the notion of state. We follow Rubin's (2001) discussion suggesting that "constitutive equations must depend on state variables that, in principle, can be measured without any prior knowledge of the past history of deformation of the material ... Within the context of this notion of state, elastic strain is a state variable, whereas the total strain and plastic strains are not state variables since they are measured with respect to an arbitrary reference configuration". Notice that this statement does not imply that the past history of the deformation is not relevant to the subsequent material behaviour. The past history can be accounted for by employing state variables such as porosity that depends on the history, but can be measured at any time without prior knowledge of that history.

The rate of change of the energy potential is then:

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \varepsilon^e_{ij}} \dot{\varepsilon}^e_{ij}. \tag{3.4}$$

The elastic strain rate is then expressed in terms of the total and relaxation strain rates:

$$\dot{\varepsilon}^e_{ij} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}^r_{ij}. \tag{3.5}$$

The relaxation strain rate can be understood to control the rate at which the elastic strain energy establishes its local minimum. In elasto-plasticity, the relaxation strain rate is equivalent to the rate of plastic straining $\dot{\varepsilon}^r_{ij} = \dot{\varepsilon}^p_{ij}$. Notice that the above decomposition in strain rates does not necessarily infer the existence of the cumulative decomposition of strains in the form $\varepsilon^e_{ij} = \varepsilon_{ij} - \varepsilon^p_{ij}$. Instead, and consistent with Rubin (2001), the current formulation produces constitutive responses that are independent on the cumulative total strain, ε_{ij}, or plastic strain, ε^p_{ij} (only on their rates).

Combining Eqs. (3.1), (3.4) and (3.5) gives:

$$\widetilde{\Phi} = \left(\sigma_{ij} - \frac{\partial \Psi}{\partial \varepsilon^e_{ij}}\right)\dot{\varepsilon}^e_{ij} + \sigma_{ij}\dot{\varepsilon}^r_{ij}. \tag{3.6}$$

Employing the standard step, stress is identified conjugated to the elastic strain (Ziegler, 1983):

$$\sigma_{ij} \equiv \sigma_{ij}\left(\varepsilon^e_{ij}\right) = \frac{\partial \Psi}{\partial \varepsilon^e_{ij}}. \tag{3.7}$$

Therefore, the rate of dissipation could be recovered from Eq. (3.6)

$$\widetilde{\Phi} = \sigma_{ij}\dot{\varepsilon}^r_{ij} \geqslant 0. \tag{3.8}$$

With the use of Eqs. (3.5) and (3.7), the stress rate is given as:

$$\dot{\sigma}_{ij} = \frac{\partial^2 \Psi}{\partial \varepsilon^e_{ij}\partial \varepsilon^e_{kl}}\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^r_{kl}\right). \tag{3.9}$$

The elasticity stiffness tensor is then identified as $E_{ijkl} \equiv E_{ijkl}(\varepsilon^e_{ij}) = \partial^2 \Psi(\varepsilon^e_{ij})/\partial \varepsilon^e_{ij}\partial \varepsilon^e_{kl}$. Given the stress and elastic strain relation (Eq. (3.7)), it is also possible to express the stiffness tensor in terms of the stress $E_{ijkl} \equiv E_{ijkl}(\sigma_{ij})$, such that Eq. (3.9) could be compared with Eq. (2.1).

In the next section the notion of yield surfaces will be introduced as a mean to determine the relaxation (or plastic) strain rate $\dot{\varepsilon}^r_{kl}$, and thus close Eq. (3.9) without violating Eq. (3.8). However, an alternative way to do that would be proposed in Section 5, revealing a hypo-plastic structure.

4. Elasto-plastic and hyper-plastic models

To complete an elasto-plastic model formulation, it is possible to distinguish between the elastic and elasto-plastic regions of deformation using an assumed yield surface in a combined stress and elastic strain space

$$y_\Phi\left(\varepsilon^e_{ij}, \sigma_{ij}\right) \leqslant 0. \tag{4.1}$$

Indeed, the elastic strain is related directly to the stress through $\varepsilon^e_{ij} \equiv \varepsilon^e_{ij}(\sigma_{ij})$. This is consistent with Collins and Houlsby (1997), who defined the yield function as $y_\Phi\left(\chi_{ij}, \sigma_{ij}, \varepsilon^p_{ij}\right) \leqslant 0$ in terms of the generalised stress $\chi_{ij} \equiv \sigma_{ij}\left(\varepsilon^e_{ij}\right) - \rho_{ij}\left(\varepsilon^p_{ij}\right)$. The use of both σ_{ij} and χ_{ij} in y_Φ helped Collins and Houlsby to define non-associated flow rules. Similar non-associated flow rule could be developed on the basis of Eq. (4.1), which removes any dependence on cumulative plastic strain, either directly or indirectly (through the shift stress ρ_{ij}).

The use of subscript 'Φ' in y_Φ highlights that this yield function must be chosen in a way that guarantees non-negative dissipation, $\widetilde{\Phi} \geqslant 0$ (Eq. (4.8)). The relaxation strain rate $\dot{\varepsilon}^r_{ij}$ is defined by the plastic strain rate $\dot{\varepsilon}^p_{ij}$, which is associated to the gradient of the assumed yield surface

$$\dot{\varepsilon}^r_{ij} = \dot{\varepsilon}^p_{ij} = \lambda_\Phi \frac{\partial y_\Phi}{\partial \sigma_{ij}} \tag{4.2}$$

where λ_Φ is the non-negative plasticity multiplier, for which a constitutive equation will be provided later. Consulting with Eq. (3.8), the non-negative dissipation condition can be ensured as long as y_Φ is a general star-shape, containing its origin and in the stress space (Ziegler, 1983; Hunter, 1983):

$$\frac{\partial y_\Phi}{\partial \sigma_{ij}}\sigma_{ij} \geqslant 0 \tag{4.3}$$

It is possible to replace the elastic strain with the stress using Eq. (3.7), such that in stress space:

$$y(\sigma_{ij}) = y_\Phi\left(\varepsilon^e_{ij}(\sigma_{ij}), \sigma_{ij}\right) \leqslant 0 \tag{4.4}$$

Notice, however, that the flow rule does not necessarily have to be associated to the yield surface in the stress space, $y(\sigma_{ij})$, since:

$$\dot{\varepsilon}^p_{ij} = \lambda_\Phi\left(\frac{\partial y}{\partial \sigma_{ij}} - \frac{\partial y_\Phi}{\partial \varepsilon^e_{kl}}\frac{\partial \varepsilon^e_{kl}(\sigma_{ij})}{\partial \sigma_{ij}}\right) = \lambda_\Phi\left(\frac{\partial y}{\partial \sigma_{ij}} - \frac{\partial y_\Phi}{\partial \varepsilon^e_{kl}}E^{-1}_{klij}\right) \tag{4.5}$$

Associated flow rules in the stress space are therefore related to y_Φ independently of the elastic strain. This result is consistent with Collins and Houlsby (1997). Nevertheless, as mentioned above we have avoided introducing the dependence of y_Φ on cumulative plastic or total strain, which are both reference dependent. Compared to Collins and Houlsby (1997) this is rather more limiting, but, again, is made consistent with Rubin's (2001) discussion on state variables. Many elasto-plastic models use yield functions dependent on the cumulative plastic strain as a mean to model isotropic hardening. Mathematically, it is straightforward to include those dependencies in the current formulation in a way that will enable the description of isotropic hardening, but it is more physical to construct models dependent on other history dependent internal variables that are reference-free, e.g., porosity as treated by Rubin et al. (1996), the reference-free breakage in Einav (2007), their combination in Rubin and Einav (in press), and microstrain in Tjahjanto et al. (2008). For the sake of brevity, introducing such variables is a topic for future developments, which would demonstrate the ability of the current formulation to address the issue of isotropic hardening in a way more physical than adding yield dependence on reference-dependent plastic strains.

The constitutive equation for the non-negative multiplier λ_Φ can be found from the consistency condition of the yield function ($\dot{y} = \dot{y}_\Phi = 0$) and the flow rule Eq. (4.2),

$$\lambda_\Phi\left(\varepsilon^e_{ij}(\sigma_{ij}),\sigma_{ij},\dot{\varepsilon}_{ij}\right) = \lambda(\sigma_{ij},\dot{\varepsilon}_{ij}) = \frac{\frac{\partial y}{\partial \sigma_{ij}}\frac{\partial^2 \Psi}{\partial \varepsilon^e_{ij}\partial \varepsilon^e_{kl}}}{\frac{\partial y}{\partial \sigma_{mn}}\frac{\partial^2 \Psi}{\partial \varepsilon^e_{mn}\partial \varepsilon^e_{pq}}\frac{\partial y_\Phi}{\partial \sigma_{pq}}}\dot{\varepsilon}_{kl} \tag{4.6}$$

which should be solved coupled to the Kuhn–Tucker optimality conditions in the form

$$\lambda_\Phi \geqslant 0; \quad y_\Phi \leqslant 0; \quad \lambda_\Phi y_\Phi = 0. \tag{4.7}$$

Those conditions ensure distinct separation between purely elastic and elasto-plastic regimes (Eqs. (4.1) and (4.2)), and satisfy the constraint given by Eq. (4.4).

Next, let us define the following singularity functions:

$$I(x) = \{0 \quad \text{if } x < 0, 1 \quad \text{if } x = 0, \quad \text{empty for } x > 0\} \tag{4.8}$$

$$\langle x\rangle = \{0 \quad \text{if } x < 0, x \quad \text{if } x \geqslant 0\} \tag{4.9}$$

where the first relates to the indicator function; the second is known as the Macaulay bracket function. It is emphasized that through the foregoing use of the consistency condition it is ensured that the yield function does not become positive; therefore, $I(y)$ does not have to be specified for values $y = y_\Phi > 0$. This is the reason for the use of 'empty' in the definition of $I(x)$ for $x > 0$. Using Eq. (3.9), the complete stress–strain rate equation of hyper-plastic models may, in fact, be represented through a single constitutive equation that satisfies the Kuhn–Tucker optimality conditions

$$\dot{\sigma}_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij}) = \frac{\partial^2 \Psi}{\partial \varepsilon^e_{ij}\partial \varepsilon^e_{kl}}\left(\dot{\varepsilon}_{kl} - \langle\lambda\rangle I(y)\frac{\partial y_\Phi}{\partial \sigma_{kl}}\right) \tag{4.10}$$

where $I(y) = I(y_\Phi)$. Notice that the above elasto-plasticity formulation is limited to describing idealised rate-independent processes, since $\lambda(\sigma_{ij},\dot{\varepsilon}_{ij}dt) = \lambda(\sigma_{ij},\dot{\varepsilon}_{ij})dt, \forall dt > 0$. The models satisfy the thermodynamics restrictions in Section 3; therefore, they can be classified as hyper-plastic.

5. Thermodynamic hypo-plastic models: h^2plasticity

The hyper-plastic formulation (Section 4) is only one of several formulations that may be structured to satisfy the laws of thermodynamics. Another, more general formulation is developed in this section that reveals a hypo-plastic structure.

5.1. General impositions

Consider all the thermodynamics foundations in Section 3, and assume the following relaxation strain rate tensor to close Eq. (3.9):

$$\dot{\varepsilon}^r_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij}) = \Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})f_{ij}(\sigma_{ij}), \quad \text{with } \Gamma(\sigma_{ij},\dot{\varepsilon}_{ij}) \geqslant 0. \tag{5.1}$$

Given the positiveness of $\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})$, the condition in Eq. (2.1b) is automatically satisfied. Rate independence further requires selecting $\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})$ that would satisfy Eq. (2.2). According to Eq. (3.8) the dissipation now reads as follows:

$$\widetilde{\Phi} = \Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})\sigma_{ij}f_{ij}(\sigma_{ij}) \tag{5.2}$$

Therefore, since the scalar function $\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})$ was defined non-negative, the dissipation $\widetilde{\Phi}$ would also be non-negative, as long as the 'flow function' $f_{ij} \equiv f_{ij}(\sigma_{ij})$ is specified such that $f_{ij}\sigma_{ij} \geqslant 0$. This condition can be met by defining a scalar potential $A \equiv A(\varepsilon^e_{ij},\sigma_{ij})$ (here termed the 'acoustic potential function'), from which $f_{ij} \equiv f_{ij}(\sigma_{ij})$ is derived:

$$f^*_{ij}\left(\varepsilon^e_{ij},\sigma_{ij}\right) = \frac{\partial A\left(\varepsilon^e_{ij},\sigma_{ij}\right)}{\partial \sigma_{ij}}, \quad \text{then} \quad f_{ij} \equiv f_{ij}(\sigma_{ij}) = f^*_{ij}\left(\varepsilon^e_{ij}(\sigma_{ij}),\sigma_{ij}\right) \tag{5.3}$$

The use of 'acoustic' is chosen to highlight connection between the new potential A to acoustic emission. Since f^*_{ij} is given by the partial derivative of $A\left(\varepsilon^e_{ij},\sigma_{ij}\right)$, only with respect to σ_{ij}, A may better be viewed as a pseudo-potential, which ensures non-negative dissipation as long as it satisfies the following condition:

$$\widetilde{\Phi} = \Gamma\sigma_{ij}\frac{\partial A}{\partial \sigma_{ij}} \geqslant 0, \quad \text{thus } \frac{\partial A}{\partial \sigma_{ij}}\sigma_{ij} \geqslant 0. \tag{5.4}$$

Notice the similarities with the hyper-plastic Eqs. (4.2) and (4.3). The role of the yield surface in hyper-plasticity is being replaced by the acoustic potential in the current formulation, which does not distinctly separate between purely elastic and purely elasto-plastic regimes.

Next, combining Eq. (3.9) with Eqs. (5.1) and (5.3) gives:

$$\dot{\sigma}_{ij} = \frac{\partial^2 \Psi}{\partial \varepsilon^e_{ij}\partial \varepsilon^e_{kl}}\left(\dot{\varepsilon}_{kl} - \Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})\frac{\partial A}{\partial \sigma_{ij}}\right). \tag{5.5}$$

with $\Gamma \geqslant 0$. Given the conjugality between the stress and the elastic strain (Eq. (3.7)), the above equation satisfies the hypo-plastic structure mentioned in Eq. (2.1). The major difference is however, that the current equation was based on the thermodynamics formulation in Section 3. Therefore, the current hypo-plastic models ensure the energy conservation via Eq. (3.1). The tensor E_{ijkl} is derivable from the Helmholtz free energy potential Ψ, just as in hyper-plasticity; and finally, the convexity of A in stress space and the non-negative rate function Γ always guarantee having a non-negative dissipation $\widetilde{\Phi} = \Gamma\sigma_{ij}(\partial A/\partial \sigma_{ij}) \geqslant 0$.

5.2. A simple but robust example of the acoustic potential

The acoustic potential function can be made to share the structure of hyper-plastic yield functions $y_\Phi \leqslant 0$, by specifying $A \equiv A(y_\Phi)$. For that purpose, Eq. (5.4) requires that

$$\frac{\partial A}{\partial \sigma_{ij}}\sigma_{ij} = \frac{\partial A}{\partial y_\Phi}\frac{\partial y_\Phi}{\partial \sigma_{ij}}\sigma_{ij} \geqslant 0. \tag{5.6}$$

The rate of dissipation will thus be strictly non-negative as long as the acoustic potential monotonously increases with y_Φ:

$$\frac{\partial A}{\partial y_\Phi} \geqslant 0. \tag{5.7}$$

For example, consider the following canonical form of hyper-plastic yield functions:

$$y_\Phi\left(\varepsilon^e_{ij},\sigma_{ij}\right) = \gamma\left(\varepsilon^e_{ij},\sigma_{ij}\right) - 1 \leqslant 0, \quad \text{with } \gamma\left(\varepsilon^e_{ij},\sigma_{ij}\right) \geqslant 0 \tag{5.8}$$

with γ and y_Φ being non-dimensional. Next, assume a simple, yet robust acoustic potential:

$$A(y_\Phi) = \frac{\gamma^{s+1}}{s+1} = \frac{(1+y_\Phi)^{s+1}}{s+1}, \quad s > 0 \tag{5.9}$$

Such that

$$\frac{\partial A}{\partial y_\Phi} = (1+y_\Phi)^s, \quad s > 0 \tag{5.10}$$

$$\frac{\partial A}{\partial y_\Phi} = I(y_\Phi), \quad \text{when } s \to \infty \tag{5.11}$$

which satisfy the requirement imposed by Eq. (5.7).

5.3. Non-negative function and incremental response

To complete the formulation all that is left is to define an appropriate form of positively definite function Γ. For rate-independent materials, this function must satisfy the requirement:

$$\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij}dt)=\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})dt \quad \forall dt>0, \tag{5.12}$$

such that $\dot{\sigma}_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij}dt)=\dot{\sigma}_{ij}(\sigma_{ij},\dot{\varepsilon}_{ij})dt,\ \forall dt>0$.

One way to guarantee meeting these two requirements can be considered by taking:

$$\Gamma=\|\dot{\varepsilon}_{ij}\|\geqslant 0 \tag{5.13}$$

This, together with Eq. (5.5) represents a thermodynamically admissible hypo-plastic constitutive law that follows the popular hypo-plastic model structure mentioned by Eq. (2.3).

Another way can follow by employing the multiplier λ as the basis for the choice of Γ, in such a way that it will carry the tools established for the hyper-plasticity formulation to handle imposed failure criteria (as shown by Eq. (4.6)). The first option to achieve this can be defined by:

$$\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})=\langle\lambda(\sigma_{ij},\dot{\varepsilon}_{ij})\rangle\geqslant 0. \tag{5.14}$$

In this way, the current formulation provides a rather broad class of rate independent (h^2plastic) models that satisfy the laws of thermodynamics:

$$\dot{\sigma}_{ij}=\frac{\partial^2\Psi}{\partial\varepsilon^e_{ij}\partial\varepsilon^e_{kl}}\left(\dot{\varepsilon}_{kl}-\langle\lambda\rangle(1+y)^s\frac{\partial y_\Phi}{\partial\sigma_{kl}}\right),\quad \forall s>0 \tag{5.15a}$$

$$\dot{\sigma}_{ij}=\frac{\partial^2\Psi}{\partial\varepsilon^e_{ij}\partial\varepsilon^e_{kl}}\left(\dot{\varepsilon}_{kl}-\langle\lambda\rangle I(y)\frac{\partial y_\Phi}{\partial\sigma_{kl}}\right),\quad \text{if } s\to\infty \tag{5.15b}$$

For $s\to\infty$, the above equation is identical to the hyper-plastic constitutive equation given by Eq. (4.10). However, upon using a finite value of $s>0$ the current constitutive equation predicts a continuous variation in the tangential stiffness. Furthermore, irrespective to s, as the loading proceeds towards $y=0$, both $I(y)$ and $(1+y)^s$ equals unity, such that the relaxation strain rate $\dot{\varepsilon}^r_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij})$ is identical to $\dot{\varepsilon}^p_{kl}(\sigma_{ij},\dot{\varepsilon}_{ij})$ of hyper-plasticity. This means that at this point the two formulations behave identically in terms of the response and failure determination.

Another way to facilitate the construction of models with a hyper-plastic structure at the limit $s\to\infty$, can for example be given by replacing Eq. (5.14) with:

$$\Gamma(\sigma_{ij},\dot{\varepsilon}_{ij})=|\lambda(\sigma_{ij},\dot{\varepsilon}_{ij})|\geqslant 0. \tag{5.16}$$

In this case, the constitutive equation is different from that given by Eq. (5.15) merely through the term $\langle\lambda\rangle$ being replaced with $|\lambda|$. In the case of $s\to\infty$ the solution is, in fact, practically identical to hyper-plasticity (i.e. using Eq. (4.10)). If at time t, $y(t)<0$ the response is still elastic. Similarly, the response is elasto-plastic if $y(t)=0$ and $\lambda\ (t)\geqslant 0$. However, if at time t, $y(t)=0$ and $\lambda(t)<0$, $\Gamma(t)$ would strictly be positive such that momentarily the response is not elastic since $\dot{\varepsilon}^r_{kl}(t)$ does not strictly vanish. However, at time $t+dt$, $I(y(t+dt))=0$, thus $\dot{\varepsilon}^r_{kl}(t+dt)$ vanishes and ensures a purely elastic response. Since we may choose dt as small as we wish, the use of 5.17 instead of (5.15) still admits a hyper-plastic model structure at the $s\to\infty$ extreme.

6. Constructing a simple 1D unifying model of hyper and hypo plasticity

6.1. Step I: Defining hyper-plastic spring-slider model

Recall the particular Helmholtz free energy potential and yield function that defines the spring-slider model (i.e. elastic perfectly-plastic 1D model):

$$\Psi=\frac{1}{2}E\varepsilon_e^2 \tag{6.1}$$

$$y_\Phi=y=\frac{|\sigma|}{k}-1\leqslant 0 \tag{6.2}$$

with k being the slider threshold. The stress and its rates are then given by

$$\sigma=\frac{\partial\Psi}{\partial\varepsilon_e}=E\varepsilon_e \tag{6.3}$$

$$\dot{\sigma}=E\dot{\varepsilon}_e=E(\dot{\varepsilon}-\dot{\varepsilon}_p) \tag{6.4}$$

where E denotes the spring Young's modulus. In hyper-plasticity the plastic strain rate is given from the flow rule:

$$\dot{\varepsilon}_p=\lambda_\Phi\frac{\partial y_\Phi}{\partial\sigma}=\lambda_\Phi\frac{sgn(\sigma)}{k} \tag{6.5}$$

where, λ_Φ, the non-negative multiplier can be found from the consistency condition:

$$\dot{y}=\frac{sgn(\sigma)}{k}\dot{\sigma}=\frac{E}{k}\left(sgn(\sigma)\dot{\varepsilon}-\frac{\lambda_\Phi}{k}\right)=0 \tag{6.6}$$

where use has been made with Eqs. (6.4) and (6.5). This equation can now be solved for λ_Φ:

$$\lambda_\Phi=\lambda(\sigma,\dot{\varepsilon})=ksgn(\sigma)\dot{\varepsilon} \tag{6.7}$$

6.2. Step II: Extension to h^2plastic model

Considering Eq. (5.16), it is now possible to define the h^2 plastic non-negative function Γ:

$$\Gamma(\sigma,\dot{\varepsilon})=|\lambda(\sigma,\dot{\varepsilon})|=k|\dot{\varepsilon}|\geqslant 0. \tag{6.8}$$

Recalling Eqs. (5.9) and (5.3), the following acoustic potential and flow functions are specified:

$$A(\sigma)=\frac{1}{s+1}\left(\frac{|\sigma|}{k}\right)^{s+1},\quad \forall s>0 \tag{6.9}$$

$$f(\sigma)=\frac{\partial A}{\partial\sigma}=\frac{sgn(\sigma)}{k}\left(\frac{|\sigma|}{k}\right)^s \tag{6.10}$$

To complete an h^2 plastic 1D model, Eq. (5.1) is used to generalise the constitutive equation for the plastic strain rate:

$$\dot{\varepsilon}_r(\sigma,\dot{\varepsilon})=\Gamma(\sigma,\dot{\varepsilon})f(\sigma)=|\dot{\varepsilon}|sgn(\sigma)\left(\frac{|\sigma|}{k}\right)^s \tag{6.11}$$

Therefore, it is readily shown that the stress–strain rate equation of this h^2 plastic model retains a hypo-plastic model form:

$$\dot{\sigma}=E\left(\dot{\varepsilon}-|\dot{\varepsilon}|sgn(\sigma)\left(\frac{|\sigma|}{k}\right)^s\right),\quad \forall s>0 \tag{6.12}$$

Interestingly, a similar 1D hypo-plastic model has been proposed previously by Wu and Kolymbas (2000), although without considering situations with negative stress σ. The model can be rewritten as follows:

$$\dot{\sigma}=E(1-sgn(\widetilde{W})(y+1)^s)\dot{\varepsilon},\quad \forall s>0 \tag{6.13a}$$

$$\dot{\sigma}=E(1-sgn(\widetilde{W})I(y))\dot{\varepsilon},\quad s\to\infty \tag{6.13b}$$

where $\widetilde{W}=\sigma\dot{\varepsilon}$ denotes the mechanical work in this model. For finite values of s, the current h^2 plastic model predicts a thermodynamical hypo-plastic response given by Eq. (6.12), with continuous change of the tangential stiffness. On the other hand, in the singular case of $s\to\infty$ the same model predicts a thermodynamical elasto-plastic (i.e. hyper-plastic) spring-slider response, with two distinct tangential stiffness values: E if either $y(t)<0$ or $y(t)=0$ but $\widetilde{W}<0$; and 0 otherwise (i.e. perfectly plastic). Furthermore, Eq. (6.12) shows that under monotonic loading, $|\sigma|$ will grow

asymptotically towards k, retaining the role of strength as in the case of the elasto-plastic spring-slider model. The parameter s controls the rate at which the strength is fully mobilised.

6.3. Model response under monotonic and cyclic loadings

In the following the model response is explored in relation to both monotonic and cyclic loading. Assume the initial value of $|\sigma(t=0)| < k$. Consider the following non-dimensional stress and strain:

$$\sigma^* = \sigma/k, \quad \varepsilon^* = E\varepsilon/k \tag{6.14}$$

Using Eq. (6.12) and the fact that during monotonic loading $sgn(\widetilde{W}) > 0$, it is possible to deduce the ordinary differential equation $d\sigma^* = (1 - |\sigma^*(\varepsilon)|^s)d\varepsilon^*$, which has a general solution if $\sigma(\varepsilon = 0) \equiv 0 : \varepsilon^*(\sigma^*) = \frac{1}{s}B((\sigma^*)^s, \frac{1}{s})$, with $B(a,b) = \int_0^a t^{1-b}(1-t)^{-1}dt$ being the incomplete beta function. For example, when $s = 1$ it is easy to show that $\sigma^*(\varepsilon^*) = 1 - \exp(-\varepsilon^*)$.

The full transition from hypo-plasticity to hyper-plasticity (*viz.* elasto-plasticity) can therefore be depicted analytically in Fig. 2 for the monotonic loading condition, assuming increasing value of s towards infinity. Complimentary demonstration of the performance under cyclic loading conditions is given numerically in Fig. 3. In particular, strong ratcheting is shown for small s, which vanishes with increasing s. Ratcheting is a well known phenomenon shown by many materials during cyclic loading, which was demonstrated using discrete element simulations of granular materials (Alonso-Marroquín et al., 2005). The extent of ratcheting revealed by hypo-plasticity is usually much larger than observed experimentally; on the other hand, models of elasto- and hyper-plasticity do not experience ratcheting. In combination, however, these two formulations can now enable us to utilise developments that until now were exclusive to each of these formulations, and resolve the ratcheting issue. For example, in a well-know extension of elasto-plasticity multiple yield surfaces are often defined. In Einav and Collins (2008) such a multiple surface model was established that has a clear physical meaning associated with statistical homogenisation, and thermodynamics. Combining this model with the current h^2plastic formulation offers one way of resolving the issue of ratcheting; this is a topic for a future paper, although preliminary studies have already carried out by the author and show immense promise. This figure also illustrate the constitutive response dependence on the choice of the non-negative multiplier, $\Gamma = |\lambda|$ or $\Gamma = \langle\lambda\rangle$ (followed simply by replacing Eq. (6.8)). In particular, as discussed in Section 5.3, either of these choices becomes consistent with hyper-plasticity as s increases towards infinity, in this particular model producing a spring-slider model response. One possible criticism on hypo-plastic models is that they do not in general recover an elastic response upon stress-reversal. Here, the choice of $\Gamma = \langle\lambda\rangle$ does produce an elastic response in such scenarios, which can be a possible resolution for this topic.

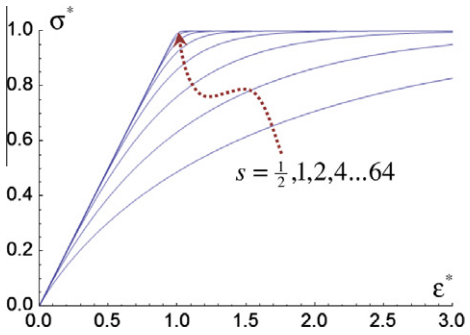

Fig. 2. Closed form stress–strain curves during monotonic loading condition with increasing s, showing transition from hypo-plasticity to hyper-plasticity.

6.4. Dissipation, work, acoustic potential, and yield

It is interesting to note the connection between the dissipation rate, the work, the acoustic potential and yield by combining Eqs. (3.8) and (6.11) (or simply through Eq. (5.2)):

$$\frac{\widetilde{\Phi}}{|\widetilde{W}|} = \frac{\partial A}{\partial y_\Phi} = (y+1)^s \geqslant 0, \quad \forall s > 0 \tag{6.15a}$$

$$\frac{\widetilde{\Phi}}{|\widetilde{W}|} = \frac{\partial A}{\partial y_\Phi} = I(y) \geqslant 0, \quad s \to \infty. \tag{6.15b}$$

In the latter singular case the model develops no dissipation as long as the material is purely in its elastic state (i.e. within the yield surface). However, upon yielding all of the work dissipates, which is a property of a spring-slider model.

The energetics given by Eqs. 6.15a,b are illustrated in Fig. 4, which highlights the role of the parameter s; in particular, the figure demonstrates that hyper-plasticity is the singular case of hypo-plasticity when $s \to \infty$. In the spirit of Gudehus's (2006) view on hypo-plasticity, the parameter s can be seen as a seismic parameter related to acoustic emission. The smaller it is the material shows higher dissipation at lower stresses, which could be related to seismic-like events at the smaller scale displayed by acoustic emission (e.g., from 'intermittent jumps', Bagnold, 1996; grain 'jiggling', Jiang and Liu, 2007; and particle crushing, Einav, 2007). In other words, it is possible that the value of s (or more generally, the acoustic potential A) may be evaluated directly from acoustic emission measurements such as those adopted successfully for many materials, including for granular materials (Fernandes et al., 2004), rocks (Baud et al., 2004), and for various composites (Bussiba et al., 2008), to name a few.

7. A simple h^2plastic model of von Mises type

This section illustrates an h^2 plastic stress–strain model in general tensorial space, through adaptation of the von Mises hyper-plastic model. In particular, it will be shown that it is possible to preserve the qualities shown by the former 1D h^2plastic model. It will further be shown that upon failure the dissipation given by the h^2plastic model matches that given by the hyper-plastic model, so that the consistency of the failure criterion is ensured in ease.

7.1. Step I: Defining a hyper-plastic model of von Mises type

As was illustrated by many, it is possible to derive the elastic perfectly-plastic von Mises model from thermodynamics potentials, which is thus hyper-plastic in nature. In this case, it can be defined given the following two functions:

$$\Psi = \frac{1}{2}K\varepsilon^e_{ii}\varepsilon^e_{jj} + Ge^e_{ij}e^e_{ij} \tag{7.1}$$

$$y_\Phi\left(\varepsilon^e_{ij}, \sigma_{ij}\right) = y(\sigma_{ij}) = \frac{\sqrt{\sigma'_{ij}\sigma'_{ij}}}{\sqrt{2}k} - 1 \leqslant 0 \tag{7.2}$$

where k represents the material strength in simple shear; K and G are the bulk and shear modulus; $\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$ is the deviatoric stress with δ_{ij} being the Kronecker delta (=1 for $i = j$, else = 0). The constitutive equations for the stress, the stress rate, the plastic strain rate, and the consistency condition are given from these two functions:

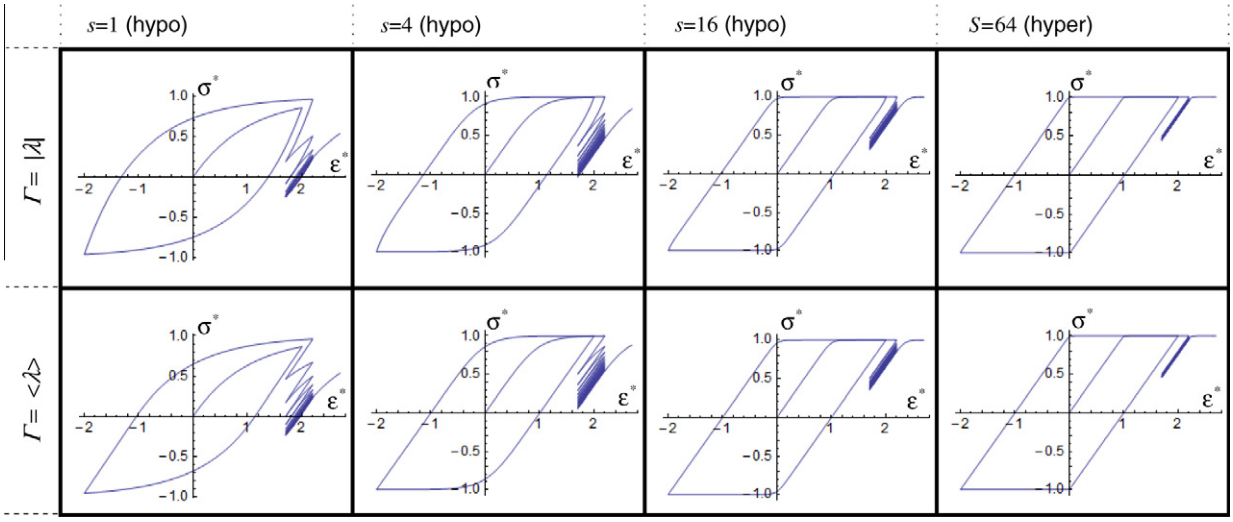

Fig. 3. Stress–strain curves during a strain controlled cycle, followed by strain controlled ratcheting experiment for different value of s. As $s \to \infty$ the well-known spring-slider element behaviour is presented, where ratcheting is eliminated.

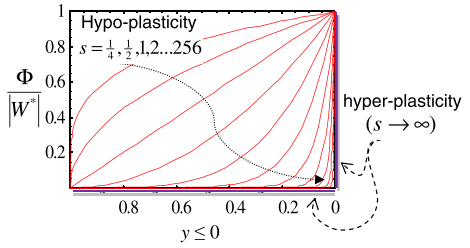

Fig. 4. Energetics of h^2plasticity, from hypo to hyper plasticity. The figure applies to 1D h^2plastic or to general von Mises h^2plastic models with $W^* = \sigma\dot{\varepsilon}$ or $W^* = q\dot{\varepsilon}_s$, respectively.

$$\sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon^e_{ij}} = K\varepsilon^e_{kk}\delta_{ij} + 2Ge^e_{ij} \Rightarrow \left\{ \sigma'_{ij} = 2Ge^e_{ij} \ \text{ and } \sigma_{kk} = 3K\varepsilon^e_{kk} \right\} \tag{7.3}$$

$$\dot{\sigma}_{ij} = \left(\left(K - \frac{2G}{3} \right)\delta_{ij}\delta_{kl} + 2G\delta_{ik}\delta_{jl} \right)\dot{\varepsilon}^e_{kl} \Rightarrow \left\{ \dot{\sigma}'_{ij} = 2G\dot{e}^e_{ij} \ \text{ and } \dot{\sigma}_{kk} = 3K\dot{\varepsilon}^e_{kk} \right\} \tag{7.4}$$

$$\dot{\varepsilon}^p_{ij} = \lambda \frac{\partial y_\Phi}{\partial \sigma_{ij}} \Rightarrow \dot{\varepsilon}^p_{ii} = 0; \quad \dot{e}^p_{ij} = \lambda \frac{S_{ij}\left(\sigma'_{ij}\right)}{\sqrt{2}k} \tag{7.5}$$

$$\dot{y}(\sigma_{ij}) = \frac{S_{ij}\left(\sigma'_{ij}\right)}{\sqrt{2}k}\dot{\sigma}'_{ij} = 0 \tag{7.6}$$

where the modified signum function is introduced (e.g., Houlsby and Puzrin, 2006) for any second order tensor a_{kl}

$$S_{ij}(a_{ij}) = \frac{a_{ij}}{\sqrt{a_{kl}a_{kl}}}. \tag{7.7}$$

The constitutive equation for the plastic multiplier is given by combining Eqs. (7.4) and (7.5):

$$\lambda\left(\sigma'_{ij}, \dot{e}_{ij}\right) = \sqrt{2}kS_{ij}\left(\sigma'_{ij}\right)\dot{e}_{ij} \tag{7.8}$$

where $\dot{e}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}$. Since the dissipation in this model is defined by $\widetilde{\Phi} = \sigma'_{ij}\dot{e}^p_{ij}$, during active yield/failure it attains the following form (i.e. when $y(\sigma_{ij}) = 0$ and $\lambda(\sigma'_{ij}, \dot{e}_{ij}) \geqslant 0$):

$$\widetilde{\Phi} = \widetilde{W}_d \geqslant 0 \tag{7.9}$$

such that it balances the deviatoric mechanical work, $\widetilde{W}_d$, defined as follows

$$\widetilde{W}_d = \sigma'_{ij}\dot{e}_{ij} \tag{7.10}$$

This balance is a well-known property of the von Mises failure criterion that is often termed the deviatoric work criterion, since it corresponds to plastic yielding when the deviatoric portion of the stored elastic energy reaches a critical amount.

7.2. Step II: Extension to h^2plastic model of von Mises type

It is now possible to define the function Γ, considering Eq. (5.16):

$$\Gamma\left(\sigma'_{ij}, \dot{e}_{ij}\right) = \left|\lambda\left(\sigma'_{ij}, \dot{e}_{ij}\right)\right| = \sqrt{2}k\left|S_{ij}\left(\sigma'_{ij}\right)\dot{e}_{ij}\right| \geqslant 0. \tag{7.11}$$

Recalling Eqs. (5.9) and (5.3), the acoustic potential and flow functions are specified as follows

$$A\left(\sigma'_{ij}\right) = \frac{1}{s+1}\left(\frac{\sqrt{\sigma'_{ij}\sigma'_{ij}}}{\sqrt{2}k}\right)^{s+1}, \quad \forall s > 0. \tag{7.12}$$

$$f_{ij}\left(\sigma'_{ij}\right) = \frac{\partial A}{\partial \sigma'_{ij}} = \frac{S_{ij}\left(\sigma'_{ij}\right)}{\sqrt{2}k}\left(\frac{\sqrt{\sigma'_{kl}\sigma'_{kl}}}{\sqrt{2}k}\right)^{s}. \tag{7.13}$$

To complete the h^2 plastic von Mises model, Eq. (5.1) is used to give the relaxation strain rate:

$$\dot{e}^r_{ij}\left(\sigma'_{ij}, \dot{e}_{ij}\right) = \Gamma\left(\sigma'_{ij}, \dot{e}_{ij}\right)f_{ij}\left(\sigma'_{ij}\right) = \frac{|\sigma'_{kl}\dot{e}_{kl}|}{2k^2}\left(\frac{\sqrt{\sigma'_{mn}\sigma'_{mn}}}{\sqrt{2}k}\right)^{s-2}\sigma'_{ij} \tag{7.14}$$

Therefore, the stress–strain rate equation of this h^2plastic model is readily shown to retain a hypo-plastic model form along the deviatoric plane:

$$\dot{\sigma}_{kk} = 3K\dot{\varepsilon}_{kk}; \quad \dot{\sigma}'_{ij} = 2G\left(\dot{e}_{ij} - \frac{|\sigma'_{kl}\dot{e}_{kl}|}{2k^2}\left(\frac{\sqrt{\sigma'_{mn}\sigma'_{mn}}}{\sqrt{2}k}\right)^{s-2}\sigma'_{ij}\right). \tag{7.15}$$

Since in this model plastic relaxation applies only to the deviatoric part (i.e., $\Phi = \sigma'_{ij}\dot{e}^r_{ij}$), the dissipation reads as follows

$$\widetilde{\Phi} = (1+y)^s|\widetilde{W}_d| \geqslant 0, \quad \forall s > 0 \tag{7.16}$$

which upon active yield/failure (i.e. when $y(\sigma_{ij}) = 0$ and $\lambda\left(\sigma'_{ij}, \dot{e}_{ij}\right) \geqslant 0$) matches the dissipation produced by the original hyper-plastic von Mises model (Eq. (7.9)), so that the consistency of the failure

criterion is ensured in ease. Also, the h^2plastic model is consistent with the hyper-plastic model at the limit of $s \to \infty$, since

$$\widetilde{\Phi} = I(y)\left|\widetilde{W}_d\right| \geqslant 0, \quad s \to \infty \tag{7.17}$$

i.e., with no rate of dissipation when $y(\sigma_{ij}) < 0$.

Finally, after some elaboration of Eq. (7.15b) it is possible to show that the stress–strain rate equation of the current tensorial h^2plastic model is consistent with the 1D model, by employing the following non-dimensional stress $\sigma^* = q/k$, and strain $\varepsilon^* = 3G\varepsilon_s/k$, with $q = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$, $\varepsilon_s = \sqrt{\frac{2}{3}e_{ij}e_{ij}}$, and $\widetilde{W}_d = q\dot{\varepsilon}_s$.

8. Example of a non-associated h^2plastic model

The power of the new thermodynamically consistent framework to unify hyper and hypo plastic constitutive structures is further demonstrated by examining a famous non-associative elasto-plastic model: the modified Drucker–Prager model. In the first part of this section, the model is derived within the framework of hyper-plasticity. This is done in a way similar to the derivation by Collins and Houlsby (1997), although here the model is completed with information about elasticity and without a need to define 'generalised stress' that is associated with an arbitrary reference dependent plastic strain.

It will be demonstrated again how during failure the dissipation given by the h^2plastic model matches the one given by the hyper-plastic model, so that the consistency with the failure criterion is ensured in ease, irrespective if the hyper-plastic model is associated or not.

8.1. Step I: Hyper-plastic model of Drucker–Prager type

For simplicity, let us employ the stress and strain invariants so defined at the end of the previous section. With that in mind, the Helmholtz free energy of a linear elastic hyper-plastic Drucker–Prager model takes the following form:

$$\Psi(\vec{\varepsilon}_e) = \frac{1}{2}K\varepsilon_v^{e^2} + \frac{3}{2}G\varepsilon_s^{e^2} \tag{8.1}$$

where $\vec{\varepsilon}_e = \{\varepsilon_v^e, \varepsilon_s^e\}$ denotes the elastic strain vector in this triaxial two-dimensional model. The corresponding thermodynamically conjugated (true) stresses and their rates can be derived:

$$p = \frac{\partial \Psi}{\partial \varepsilon_v^e} = K\varepsilon_v^e; \quad q = \frac{\partial \Psi}{\partial \varepsilon_s^e} = 3G\varepsilon_s^e \tag{8.2}$$

$$\dot{p} = K\dot{\varepsilon}_v^e; \quad \dot{q} = 3G\dot{\varepsilon}_s^e \tag{8.3}$$

The yield surface in mixed dissipative stress/elastic-strain space, which is the basis for the flow rule of the plastic strain rates, is taken as follows:

$$y_\Phi(\vec{\varepsilon}_e, \vec{\sigma}) = \frac{q}{\beta p + (\mu - \beta)K\varepsilon_v^e} - 1 \leqslant 0 \tag{8.4}$$

where $\vec{\sigma} = \{p, q\}$, while β and μ are parameters soon to be defined. Combining the last equation with the first relation in Eq. (8.2), the same yield surface can be written purely in stress-space:

$$y(\vec{\sigma}) = \frac{q}{\mu p} - 1 \leqslant 0. \tag{8.5}$$

which reveals the constitutive meaning of μ, being the friction coefficient.

The evolution law for the plastic strain rates is given based on the dissipative yield surface, with the use of the first relation in Eq. (8.2)

$$\dot{\varepsilon}_v^p = \lambda\frac{\partial y_\Phi}{\partial p} = -\lambda\frac{\beta}{\mu p}; \quad \dot{\varepsilon}_s^p = \lambda\frac{\partial y_\Phi}{\partial q} = \lambda\frac{1}{\mu p}. \tag{8.6}$$

The plastic dilatancy expression of this model is given as

$$d = -\frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p} = \beta, \tag{8.7}$$

which shows that β plays the role of the dilatancy coefficient. The constitutive equation for the non-negative multiplier λ is given by solving the consistency condition of the yield surface:

$$\dot{y}_\Phi(\vec{\varepsilon}_e, \vec{\sigma}) = \dot{y}(\vec{\sigma}) = \frac{1}{\mu p}(\dot{q} - \mu\dot{p}) = 0 \tag{8.8}$$

from which

$$\lambda\left(\vec{\sigma}, \dot{\vec{\varepsilon}}\right) = \mu p\dot{\varepsilon}^* \tag{8.9}$$

where the following equivalent strain is defined for this model

$$\dot{\varepsilon}^* = \frac{\dot{\varepsilon}_s - \frac{\mu K}{3G}\dot{\varepsilon}_v}{1 + \beta\frac{\mu K}{3G}} \tag{8.10}$$

Since the dissipation in this model is expressed as $\widetilde{\Phi} = p\dot{\varepsilon}_v^p + q\dot{\varepsilon}_s^p$, then during active yield/failure (i.e. when $y(\vec{\sigma}) = 0$ and $\lambda\left(\vec{\sigma}, \dot{\vec{\varepsilon}}\right) \geqslant 0$, or in other words when $y(\vec{\sigma}) = 0$ and $\dot{\varepsilon}_s \geqslant \frac{\mu K}{3G}\dot{\varepsilon}_v$):

$$\widetilde{\Phi} = p(\mu - \beta)\dot{\varepsilon}^* \geqslant 0. \tag{8.11}$$

Otherwise, there is no dissipation.

In developing the h^2plastic version of this model, we will prove that during active failure both models give the same dissipation as in Eq. (8.11), but in the h^2plastic model, dissipation develops even prior to the failure condition (i.e. even when $y(\vec{\sigma}) < 0$).

8.2. Step II: H^{2}plastic model of Drucker–Prager type

The following non-negative function may, for example, be defined via:

$$\Gamma(\vec{\sigma}, \dot{\vec{\varepsilon}}) = \left\langle \lambda(\vec{\sigma}, \dot{\vec{\varepsilon}}) \right\rangle = \mu p\langle\dot{\varepsilon}^*\rangle \geqslant 0. \tag{8.12}$$

Motivated by Eqs. (5.9) and (5.3), the following acoustic potential and flow functions are specified

$$A(\vec{\varepsilon}_e, \vec{\sigma}) = \frac{1}{s+1}\left(\frac{q}{\beta p + (\mu - \beta)K\varepsilon_v^e}\right)^{s+1}, \quad \forall s > 0. \tag{8.13}$$

$$\vec{f}(\vec{\sigma}) = \{f_p, f_q\} = \left\{\frac{\partial A}{\partial p}, \frac{\partial A}{\partial q}\right\} = \left\{-\frac{\beta}{\mu p}\left(\frac{q}{\mu p}\right)^{s+1}, \frac{1}{\mu p}\left(\frac{q}{\mu p}\right)^{s}\right\} \tag{8.14}$$

To complete the h^2 plastic model, Eq. (5.1) is used to generalise the constitutive equation for the relaxation strain rate $\left(\dot{\vec{\varepsilon}}_r(\vec{\sigma}, \dot{\vec{\varepsilon}}) = \Gamma(\vec{\sigma}, \dot{\vec{\varepsilon}})\vec{f}(\vec{\sigma})\right)$:

$$\dot{\varepsilon}_v^r(\vec{\sigma}, \dot{\vec{\varepsilon}}) = \Gamma(\vec{\sigma}, \dot{\vec{\varepsilon}})f_p(\vec{\sigma}) = -\beta\left(\frac{q}{\mu p}\right)^{s+1}\langle\dot{\varepsilon}^*\rangle;$$

$$\dot{\varepsilon}_s^r(\vec{\sigma}, \dot{\vec{\varepsilon}}) = \Gamma(\vec{\sigma}, \dot{\vec{\varepsilon}})f_q(\vec{\sigma}) = \left(\frac{q}{\mu p}\right)^{s}\langle\dot{\varepsilon}^*\rangle. \tag{8.15}$$

In this extended model the dilation angle can be calculated from Eq. (8.15):

$$d = -\frac{\dot{\varepsilon}_v^r}{\dot{\varepsilon}_s^r} = \beta\left(\frac{q}{\mu p}\right). \tag{8.16}$$

Therefore, the dilation angle retains the value given by the hyper-plastic model only when the full yield criterion is mobilised (i.e. $y = 0 \Rightarrow q = \mu p \Rightarrow d = \beta$). Also, there is no dilatancy when the material is along the isotropic compression line ($q = 0$). The stress–strain

rate equation of the current h^2plastic model shows a hypo-plastic structure in all stress–strain directions:

$$\dot{p} = K\left(\dot{\varepsilon}_v + \beta\left(\frac{q}{\mu p}\right)^{s+1}\langle\dot{\varepsilon}^*\rangle\right); \quad \dot{q} = 3G\left(\dot{\varepsilon}_s - \left(\frac{q}{\mu p}\right)^{s}\langle\dot{\varepsilon}^*\rangle\right) \tag{8.17}$$

with

$$\widetilde{\Phi} = p(\mu-\beta)(1+y)^{s+1}\langle\dot{\varepsilon}^*\rangle \geqslant 0, \quad \forall s > 0. \tag{8.18}$$

When the model is associated (i.e., when $\beta = \mu$), this model is non-dissipative yet the relaxation strain is not vanishing and the deformation is irreversible, consistent with the original associated Drucker–Prager model. Also notice that for any s

$$\widetilde{\Phi} = p(\mu-\beta)\langle\dot{\varepsilon}^*\rangle \geqslant 0, \quad \text{when } y(\vec{\sigma}) = 0 \tag{8.19}$$

which is identical to the dissipation of the hyper-plastic model in Eq. (8.11), during active failure (i.e. when $\lambda(\vec{\sigma},\dot{\vec{\varepsilon}}) \geqslant 0$ or in other words when $\dot{\varepsilon}_s \geqslant \frac{\mu K}{3G}\dot{\varepsilon}_v$). However, the current h^2plastic model reveals dissipation even before failure (i.e. $y(\vec{\sigma}) < 0$) and also during reloading from failure (i.e. $y(\vec{\sigma}) < 0$ and $\lambda(\vec{\sigma},\dot{\vec{\varepsilon}}) < 0$ or $\dot{\varepsilon}_s < \frac{\mu K}{3G}\dot{\varepsilon}_v$).

8.3. Stress-path response to shear under isochoric condition

Fig. 5 simulates the shear response of the model under isochoric loading condition ($\dot{\varepsilon}_v = 0$), for several deviator stress controlled cycles, for various seismic parameter s, and dilation angles β. In all simulations the elastic moduli have been defined by $3G = K$, with friction coefficient $\mu = 1$. It can be noticed that as the seismic parameter s is higher, the constitutive behaviour becomes that given by the elasto-plastic model, in this case given by the elasto perfectly-plastic modified Drucker–Prager model. In the case where failure is prescribed associated (i.e. $\mu = \beta = 1$), and the dilation angle is the highest, the rate of increase in the mean stress p is the highest, always consistent with the understanding provided by the classical elasto-plastic model.

8.4. Response envelopes

An effective geometrical interpretation of the constitutive response provided by any model can be represented using the so-called response envelope introduced by Gudehus (1979). We apply this method to explore the structure of the h^2plastic model specified by Eq. (8.17). The response envelope is given as the stress rate responses to all strain rate inputs of unit magnitude, with the length of a stress rate response vector representing the directional material stiffness defined by Eq. (8.17). The first term of this equation, which is linear in the strain rate, defines stress rate responses forming an ellipsoid in the principal stress rate space. In hyper-plasticity (the case where $s \to \infty$), this component denotes the elastic constitutive response prior to yield/failure, schematically shown by the continuous ellipses in Fig. 6, with a centre denoting the initial stress point (marked by the big point).

Notice, that in this particular model the non-negative multiplier was specified by $\Gamma = \langle\lambda\rangle = \mu p\langle\dot{\varepsilon}^*\rangle \geqslant 0$. Therefore, in the hypo-plastic model case defined by $s = 1$, Eq. (8.17) is nonlinear in the strain rate when $\dot{\varepsilon}^* \geqslant 0$. Subsequently, apart from when the stress state resides on the isotropic line (i.e. when $q = 0$), the response envelopes always deviate from the ellipsoid envelopes of elasticity (demonstrated by the dotted lines). However, the choice of $\Gamma = \langle\lambda\rangle$ enables to capture elastic response upon reloading when $\dot{\varepsilon}^* < 0$.

Unlike in the limit case of the hyper-plastic response, in the hypo-plastic response case the shape of the response envelopes changes with the stress state, in a continuous manner until meeting the failure line. However, in either the hyper or hypo plastic model cases, when the stress reaches the Mohr–Coulomb failure criterion ($q \to \mu p$), the response envelopes become identical again and flatten along the failure line. In the non-associative model case ($\beta = 0$), the responses in the hypo-plastic case show anomalies to the left and along constant q/p lines; in previous hypo-plastic models response envelopes were characterised by ellipses that touch (osculate) the failure line. However, similar anomalies are shown by the hyper-plastic response envelopes at the failure case. In both cases, the anomalies reflect the use of the constitutive condition,

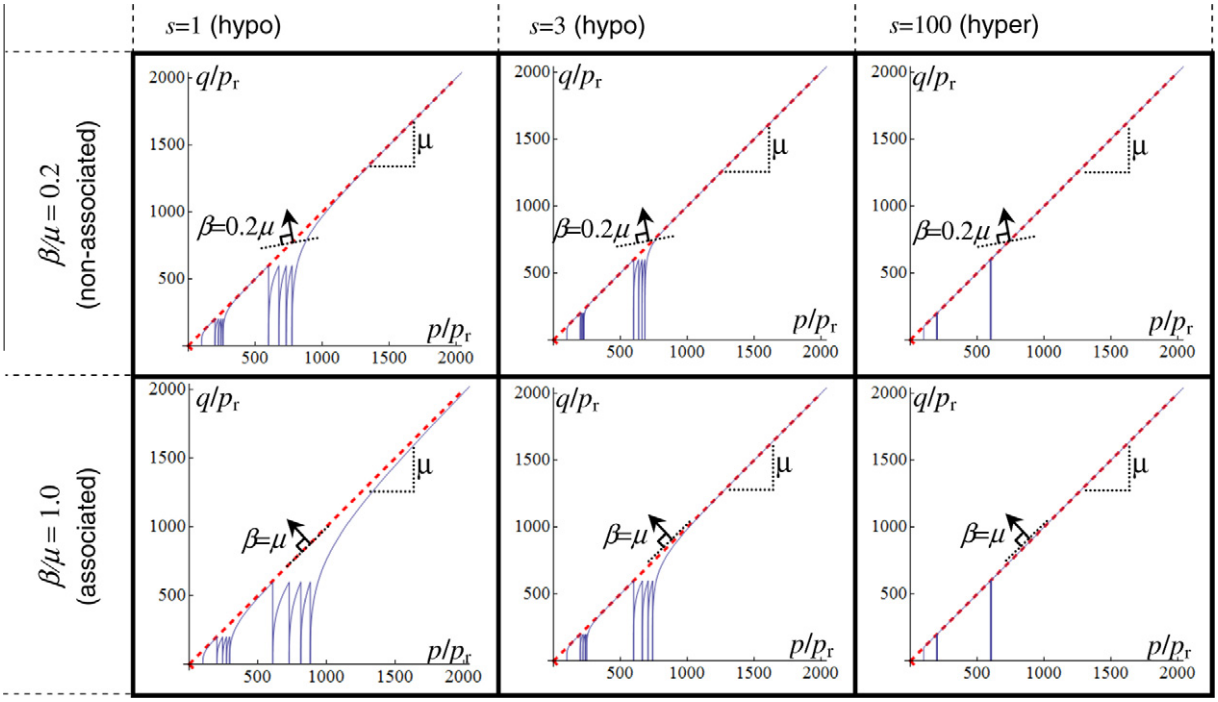

Fig. 5. Stress-path response to stress-controlled tests in h^2plasticity: from hypo-plasticity to hyper-plasticity using associated and non-associated h^2plastic Drucker–Prager model (using $\mu = 1$ and $3G = K$).

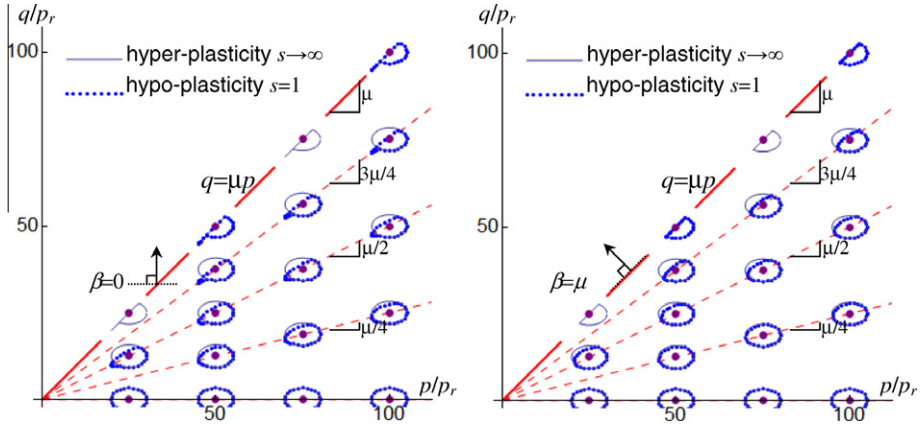

Fig. 6. Strain-rate controlled stress-response envelopes in h^2plasticity: from hypo-plasticity to hyper-plasticity using non-dilative (left plot non-associated failure rule with $\beta = 0$) and dilative (right plot, associated failure rule with $\mu = \beta = 1.0$) model cases. In both cases, the Poisson's ratio was set to 0.4. The response envelopes residing on the failure criterion (i.e. when $q = \mu p$) were drawn interchangeably between the hypo and hyper cases, to enable an easier comparison.

and the non-associative flow rules carried through to the h^2plastic model. This demonstrates that the h^2plastic formulation is indeed consistent with the notion of failure, in a way that has not been attempted by any of the previous hypo-plastic models.

9. Conclusions

A new mathematical theory has been developed that unifies together two well-established constitutive theories of plasticity: hypo-plasticity and hyper-plasticity. The unifying theory is unique in its ability to bridge between the tools created specifically to advance models under either of those previous formulations. Compared to hypo-plasticity, the new formulation can be derived from potentials, such that the laws of thermodynamics are not violated. Compared to elasto-plasticity and hyper-plasticity, the new models are both simpler for numerical implementation and superior under cyclic loading conditions, a merit revealed by showing that those models present an incrementally thoroughly non-linear constitutive structure, as in hypo-plasticity.

The fact that the new formulation can deduce thorough incremental non-linear constitutive models that satisfy the laws of thermodynamics should not be seen trivial. First, the laws of thermodynamics set one of the few (mostly) undisputable rules to protect modellers from defining absurd models. Second, through the clear definitions of both the energy potential and the non-negative rate of dissipation it is now possible to derive heat equations consistent with such hypo-plastic-like constitutive laws by associating the rate of dissipation to mechanical heat production, a standard step resulting from the use of thermodynamics principles for inelastic materials (e.g., Ziegler, 1983; Xiao et al., 2007). Third, the newly proposed 'acoustic potential' that controls the rate of dissipation continuously during the deformations hints towards a new way to calibrate the models using acoustic emission measurements (or *via* any other measurement technique from which the heat production may be evaluated, such as infrared imaging). In this way, unlike most other models, the validity of the current models can be tested independently from conventional curve-fitting using measured stress–strain data.

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