PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

*Phil. Trans. R. Soc. A* (2007) **365**, 2985–3002
doi:10.1098/rsta.2007.0009
*Published online* 13 September 2007

# Soil mechanics: breaking ground

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In soil mechanics, student's models are classified as simple models that teach us unexplained elements of behaviour; an example is the Cam clay constitutive models of critical state soil mechanics (CSSM). 'Engineer's models' are models that elaborate the theory to fit more behavioural trends; this is usually done by adding fitting parameters to the student's models. Can currently unexplained behavioural trends of soil be explained without adding fitting parameters to CSSM models, by developing alternative student's models based on modern theories?

Here I apply an alternative theory to CSSM, called 'breakage mechanics', and develop a simple student's model for sand. Its unique and distinctive feature is the use of an energy balance equation that connects grain size reduction to consumption of energy, which enables us to predict how grain size distribution (gsd) evolves—an unprecedented capability in constitutive modelling. With only four parameters, the model is physically clarifying what CSSM cannot for sand: the dependency of yielding and critical state on the initial gsd and void ratio.

**Keywords: soil mechanics; sand; constitutive modelling; critical state; breakage**

## 1. Introduction

Hand in hand to landing on the Moon, scientists from the University of Cambridge of developed the celebrated theory of critical state soil mechanics (CSSM; Roscoe & Schofield 1963; Roscoe & Burland 1968). The first successful models of this theory, the Cam clay models, were originally proposed for modelling the constitutive behaviour of clay in terms of relations among stresses, strains and void ratio. For this application, CSSM continues to provide the most elegant mathematical theory for capturing the physics of the material. In its simplicity, only five parameters are needed to describe a broad range of critical effects. However, when CSSM is adapted to model sand agglomerates, where different critical effects appear, satisfactory description requires additional fitting parameters. Certainly for extraterrestrial soils, such a phenomenological approach is doomed to fail because fitting parameters depend on extensive laboratory testing. Even on our ground, it is now well established that sand does not fit into this theory. The CSSM theory requires fitting the slope $\lambda$ of the 'isotropic-hardening' virgin compression line in the void ratio/logarithm-of-pressure space. The problem is that in sand the slope is no longer unique.

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One contribution of 20 to a Triennial Issue 'Chemistry and engineering'.

 

Another constant, the 'specific volume parameter' $N$, that with $\lambda$ sets the 'preconsolidation yielding pressure', ceases to be a constant in sand and becomes heavily dependent on the grain size distribution (gsd) and initial void ratio. Finally, CSSM predicts a critical state line (CSL) in the void ratio/logarithm-of-pressure plane, relating uniquely between the void ratio and pressure after sufficiently shearing the clay. The problem is that a unique CSL does not exist for sands owing to the material dependence on the initial gsd (Been *et al.* 1991) and history of loading (Finno & Rechenmacher 2003; Cheng *et al.* 2005). These inconsistencies are widely accepted but have not yet been overcome.

The current article presents a recent theory called 'breakage mechanics' (Einav 2007*a*,*b*) or 'breakage soil mechanics' when applied to sand. Here, I briefly develop a model based on this theory that resolves the aforementioned problems in adapting CSSM to sand. The model is a 'student's model', adopting Muir Wood's (1990) hierarchical classification, in that it is made (deliberately) simple, yet is capable of teaching us several unexplained elements of sand behaviour, already in its most simple form. It requires only four physical parameters (shear and bulk moduli, friction angle and the 'critical breakage energy') in addition to knowledge of the initial gsd and void ratio. The critical breakage energy is a new constant, which replaces the specific volume parameter $N$ and is independent of the gsd and void ratio. The result is that 'sand yielding' depends on the effective elastic moduli of the agglomerate, as well as its gsd and void ratio. This is fundamentally different to CSSM, but is justified in view of fracture mechanics and experiments. The result sets a vision for future research towards obtaining practical solutions. For the first time, constitutive models of sand can predict how the gsd evolves, extending the applicability of geotechnical constitutive models to currently unreachable problems.

## 2. The application of CSSM for modelling sand: a personal view

Constitutive relations are the cornerstone of the continuum modelling of many materials. There has been a long history of applying the theory of plasticity to soil mechanics—a theory that was adapted from applications to metal—with many scientific publications devoted to proposing new constitutive models. In reality, the hierarchy of the *widely used* plasticity models in geotechnical practice begins with the Tresca and von Mises models, then the Mohr–Coulomb model, finally ending with the Cam clay models of CSSM. Should constitutive models be complex to be used by engineers and withstand the test of time? I trust the answer is no, considering the simplicity of the aforementioned widely used plasticity models.

The ability to capture the yield transition of metals under shear deformations, so elegantly, is the basis of the success of the Tresca and von Mises elastic–plastic models. The simplicity of their corresponding yield/failure criteria made their acceptance widespread with many applications in the theory of plasticity. The next model up the hierarchy is the Mohr–Coulomb plasticity model, representing the failure limit as a linear relation between the shear stress $q$ and the pressure $p$ with a slope $M$ (i.e. in non-cohesive materials, $q = M \times p$), a constant that relates to the friction angle $\phi$ (Potts & Zdravkovic 1999). This is an important step that allowed, for the first time, to distinguish soils from metals within the confines of plasticity—but by adapting, and no longer adopting, notions from metal

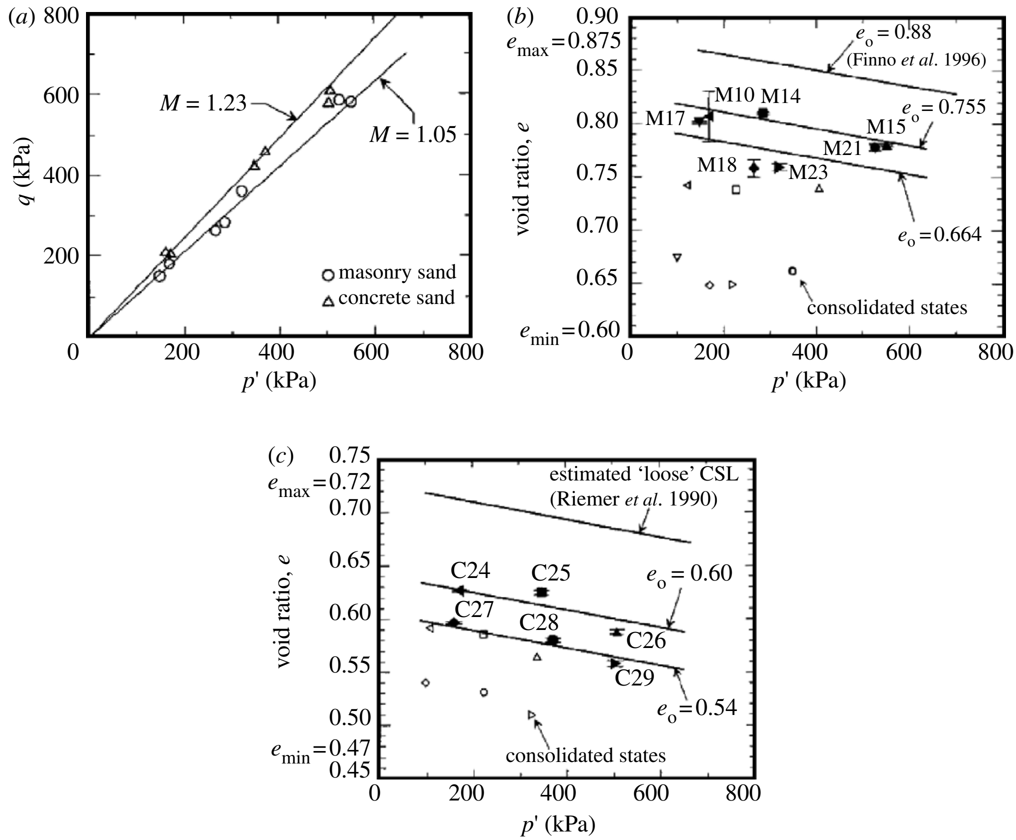


Figure 1. Ultimate conditions after consolidating masonry and concrete sands to various initial void ratios $e_0$ (but starting from similar gradings) and subjecting to drained plane strain compression tests (Finno & Rechenmacher 2003): (*a*) in shear stress/pressure space, and in void ratio/pressure space for (*b*) masonry (see Finno *et al.* 1996 for presented results) and (*c*) concrete sands (see Riemer *et al.* 1990 for presented results). The ultimate relation between shear stress and pressure is unique, but not the relation between the void ratio and pressure.

plasticity. To accommodate for the yielding of soil in compression (Terzaghi & Peck 1951), in addition to shear yielding, CSSM was originated. Unlike metal plasticity, the formulation is strongly linked to the void ratio. Initially, the Granta gravel model was developed, aimed at granular materials, but soon the idea was aborted and superseded, and attention was devoted to modelling clay (Schofield & Wroth 1968).

Unlike clay, subjecting sand to continuous shearing would not result in a unique critical state relation between void ratio and pressure (Mooney *et al.* 1998; Yamamuro & Lade 1998), although the unique relation between the shear stress and pressure is maintained (i.e. the Mohr–Coulomb $q = M \times p$). This is demonstrated in figure 1.

A possible route to allow extending CSSM to model this scenario in sand may be to introduce a family of CSLs, as applied by Daouadji & Hicher (2006) and envisaged by Muir Wood (2006). But this would need inventing at least one extra

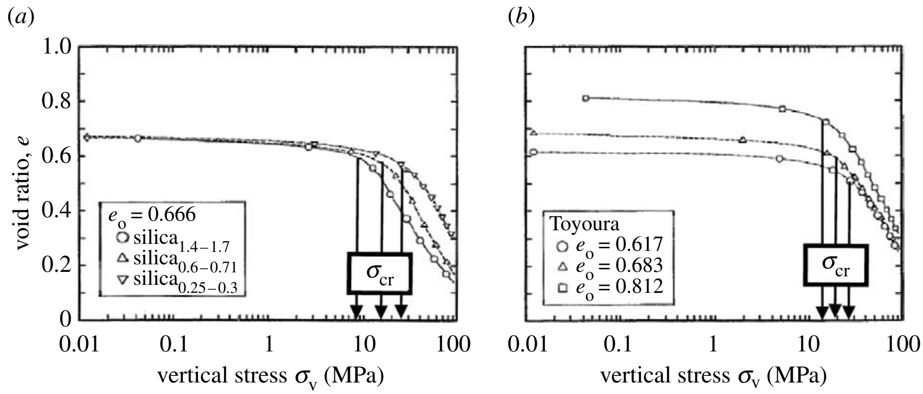


Figure 2. Effect of (*a*) initial average grain size and (*b*) void ratio $e_0$ on the initial yielding stress and $\lambda$ (the post-yield slope) in oedometer compression tests (Nakata *et al.* 2001). The corresponding vertical yielding stress is denoted as $\sigma_{cr}$, underlining relation to critical and crushing. Note that these plots are semi-logarithmic, inferring a wide range of $\sigma_{cr}$ values. Also note that all of these experiments concern only with initially poorly-graded materials (i.e. well sorted). Well-graded (poorly-sorted) materials can lead to extremely high $\sigma_{cr}$ values.

fitting parameter to span the family of lines along varying initial gradings, and probably another parameter to span the dependence on the pre-consolidation pressure. There are other difficulties involved with trying to fit CSSM to model sand. The CSSM theory requires fitting the slope $\lambda$ of the isotropic-hardening virgin compression line in the void ratio/logarithm-of-pressure space. The problem is that in sand the slope is no longer unique. Another constant, the specific volume parameter $N$, that with $\lambda$ sets the pre-consolidation yielding pressure, ceases to be a constant in sand and becomes heavily dependent on the gsd and initial void ratio (figure 2). These two additional complications mean adding two additional fitting parameters to span the range of $N$ and $\lambda$, without relating to the actual physics. All together, fitting CSSM to describe critical phenomena of sand would require inventing at least four extra artificial parameters, unlinked to the actual physical mechanisms. The Cam clay models depend on five parameters, so in total potentially bringing the toll to nine. Of course, ad hoc relations can be devised to connect between the extra parameters, but that is a fitting exercise by itself.

Models that depend on parameters that lack physical meaning may be very sensitive to the alteration of even a single physical property. Changing this property demands recalibration of the entire set of parameters (i.e. more experiments). The gsd of the soil is a perfect example of a physical property. Geotechnical engineers know for many years that changing the distribution predominantly means different behaviour; this is why the sieve analysis is probably the most common geotechnical test in practice, pivotal to soil classification. Even if correlations can be devised to obtain reliably the set of fitting parameters that are needed for adopting CSSM to modelling the behaviour of a given granular mineral, they should thereafter be recalibrated for any new given granular mineral. Therefore, in my opinion, the idea of elaborating CSSM to model the novel critical phenomena in sand by adding new fitting parameters is doomed to fail.

Now, let me ask the key question in this paper. *Can critical phenomena of sand be explained using an alternative theory, and not by adding fitting parameters that complicate CSSM models?* My personal answer is yes—with breakage (soil) mechanics.

## 3. Breakage (soil) mechanics

There have been some interesting attempts to link constitutive modelling of brittle granular materials to the actual physics of microscopic crushing (Papamichos *et al.* 1993; McDowell *et al.* 1996; Ueng & Chen 2000). These were normally based on the crushing statistics of individual particles for specific loading conditions. McDowell *et al.* (1996), for example, innovated by connecting Weibull's statistics of single grain crushing between two loading plates to the consequential increase in surface area implicitly assuming mathematically that fractal grading is ever present, though acknowledging, based on physical and numerical grounds, that this grading would emerge only at the ultimate state. While single grain crushing should have influence on the system behaviour, it is the constraining effect of the particles self-organization that governs the constitutive behaviour of the granular material to the first order. The consequence of this is that larger grains store more elastic energy than smaller particles; a fact that is fundamental to breakage mechanics.

By acknowledging that the stored energy in representative particles scales with their size, breakage mechanics (Einav 2007*a*) can consider any loading scenario. By integrating this with the concept of breakage, it is possible to link the crushing behaviour to the gsd and its evolution towards a fractal grading. The fractal grading is presented *only* after sufficiently loading the agglomerate as an ultimate grading state, and this helps replacing the CSSM postulate on the existence of the unique CSL in the void ratio/pressure space. The consequences of this will be presented in §5*c*.

The development of breakage models starts with the assumption that the current gsd could be scaled from the initial and ultimate gsd via the breakage internal variable $B$. This is different, though related to Hardin's (1985) definition of relative breakage $B_r$. An important feature is that $B$ can be measured at any time of testing, effectively by unloading the sample and subsequently sieving the particles. As portrayed in figure 3*a*, $B$ could be measured by calculating the ratio of the areas entrapped between the current, initial and ultimate gsd on the cumulative semi-logarithmic scale. Since $B$ is a variable in the formulation, it is always possible to predict how the gsd evolves.

Breakage mechanics is only briefly reviewed here, and only the end product in the form of a set of master equations is listed. The derivation of those equations may help to give a better mathematical and physical insight on what goes behind, but this can be found in the previous papers (Einav 2007*a*,*b*,*c*). The focus of this paper is the presentation of the simple student's model of breakage, setting vision for future research.

The master equations are listed only in their specialized form for triaxial test conditions; a suitable form for development of simple models. For that purpose the following notation is used: $p$, the mean effective stress; $q$, shear stress; $\varepsilon_v$, volumetric strain; and $\varepsilon_s$, shear strain.

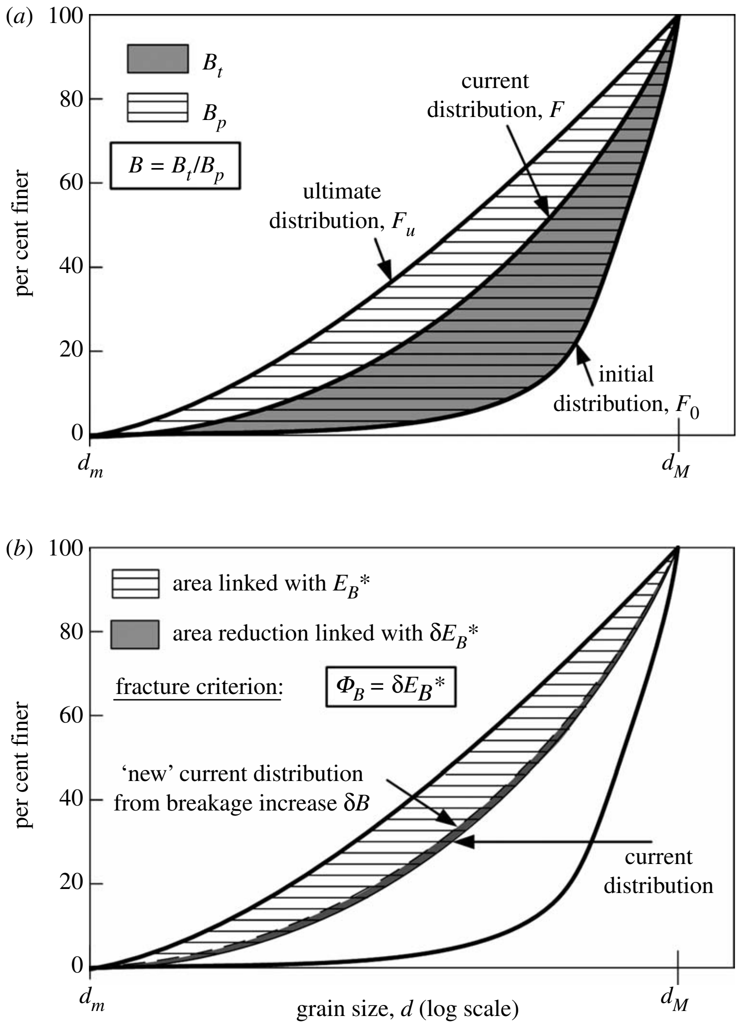


Figure 3. Breakage measurement and evolution law (Einav 2007*a*). (*a*) The measurable definition of breakage. (*b*) The breakage propagation criterion for granular materials. $\Phi_B$ is the breakage dissipation, denoting the energy consumption from incremental increase of breakage. $\delta E_B^*$ is the incremental reduction in the residual breakage energy, as defined in equation (3.5).

### (*a*) *The free energy potential of breakage models*

The reduced form of the Helmholtz free energy potential of breakage models is given by Einav (2007*a*)

$$\Psi = (1-\vartheta B)\psi_r(\varepsilon_v^e, \varepsilon_s^e), \tag{3.1}$$

where $\varepsilon_v^e$ and $\varepsilon_s^e$ are the elastic volumetric and shear strain; $\psi_r(\varepsilon_v^e, \varepsilon_s^e)$ denotes the average stored energy in a reference particle, which before any breakage ($B=0$) is simply the macroscopic free energy potential $\Psi$. Finally, $\vartheta = 1 - J_{2u}/J_{20}$ is the criticality proximity parameter, measuring how far the initial gsd is from the ultimate gsd, while $J_{20}$ and $J_{2u}$ are the second order moments of these distributions. High $\vartheta$ denotes that the initial distribution is far from the ultimate. Acknowledging that for most practical cases the ultimate gsd can be taken as fractal by mass (Sammis *et al.* 1987; McDowell *et al.* 1996), it is easy to calculate $\vartheta$, assuming that the initial gsd is given. Note that $\vartheta$ is a characteristic measure of the sand, not a parameter—just like the initial void ratio. Equation. (3.1) takes into account many important ideas, in particular, making use of a hypothesis, which was examined numerically, that the amount of energy that particles store, on average, is proportional to their surface area.

A standard thermo-mechanical analysis reveals that the stress conjugates to the kinematics variables—i.e. to the breakage $B$ and the elastic strains $\varepsilon_{\mathrm{v}}^{e}$ and $\varepsilon_{\mathrm{s}}^{e}$—are given by the corresponding derivatives (Einav 2007*a*)

$$p \equiv p(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e}, B) = \frac{\partial \Psi}{\partial \varepsilon_{\mathrm{v}}^{e}} = (1-\vartheta B)\frac{\partial \psi_{\mathrm{r}}(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e})}{\partial \varepsilon_{\mathrm{v}}^{e}}, \tag{3.2}$$

$$q \equiv q(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e}, B) = \frac{\partial \Psi}{\partial \varepsilon_{\mathrm{s}}^{e}} = (1-\vartheta B)\frac{\partial \psi_{\mathrm{r}}(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e})}{\partial \varepsilon_{\mathrm{s}}^{e}} \quad \text{and} \tag{3.3}$$

$$E_{\mathrm{B}} \equiv E_{\mathrm{B}}(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e}) = -\frac{\partial \Psi}{\partial B} = \vartheta \psi_{\mathrm{r}}(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e}) = \frac{\vartheta}{1-\vartheta B}\Psi. \tag{3.4}$$

The real stresses, $p$ and $q$, are as usual the conjugates to the (elastic) strains, and the breakage stress, $E_{\mathrm{B}}$, which is termed the 'breakage energy', is the conjugate to the breakage. The breakage energy denotes the energy that is contained in the system for breaking particles, or for shifting the gsd from the initial stage of loading, eventually leading to ultimate conditions. However, what we really like to know is how much energy is reserved in the system for breaking the particles at any given moment, i.e. from the current state to the ultimate state, and this is given by the 'residual breakage energy' $E_{\mathrm{B}}^{*}$,

$$E_{\mathrm{B}}^{*} = E_{\mathrm{B}}(1-B). \tag{3.5}$$

While the breakage energy $E_{\mathrm{B}}$ relates to the stripy area in figure 3*a*, $E_{\mathrm{B}}^{*}$ is linked to the stripy area in figure 3*b* (see Einav (2007*a*) for more detail).

### (*b*) *Breakage dissipation and breakage yield*

To derive the evolution of $B$, it is first necessary to understand how energy is dissipated from the system during an increment of breakage $\delta B$. The general form of the increment of breakage dissipation, for rate-independent processes, is given as the breakage power product between the breakage increment and its stress-like conjugate, the breakage energy $E_{\mathrm{B}}$ (Einav 2007*a*). This, combined with the postulate that the amount of breakage energy loss is directly related to the loss in the available breakage energy for crushing particles, i.e. the incremental reduction in the residual breakage energy, suggests that

$$\Phi_{\mathrm{B}} = E_{\mathrm{B}}\delta B = \delta E_{\mathrm{B}}^{*}, \tag{3.6}$$

where the increment of the residual breakage energy $\delta E_{\mathrm{B}}^{*}$ is represented schematically by the crescent grey area in figure 3*b*. The increment of the residual breakage energy is given by differentiating equation (3.5). Combining this with the last equation and integrating gives the breakage yielding criterion

$$y_{\mathrm{B}} = E_{\mathrm{B}}(1-B)^2 - E_{\mathrm{c}} \le 0, \tag{3.7}$$

where the critical breakage energy, $E_{\mathrm{c}}$, is introduced as a constant of integration (Einav 2007*c*).

The equality sign in equation (3.7) denotes active breakage growth, giving $E_{\mathrm{B}} = E_{\mathrm{c}}(1-B)^{-2}$. We can therefore choose to represent the explicit form of the dissipation (i.e. a form that is connected only to the kinematics internal variables $B$ and the $\varepsilon$'s), as $\Phi_{\mathrm{B}} = E_{\mathrm{c}}(1-B)^{-2}\delta B \ge 0$. But there is another way that helps in obtaining mathematically more favourable form of equations down the track. First, equation (3.6) can be written as $\Phi_{\mathrm{B}} = E_{\mathrm{B}}\delta B = \sqrt{E_{\mathrm{B}} E_{\mathrm{B}}(\varepsilon_{\mathrm{v}}^{\mathrm{e}}, \varepsilon_{\mathrm{s}}^{\mathrm{e}})}\delta B$. Then, exactly as before, we can use the breakage-yield condition $E_{\mathrm{B}} = E_{\mathrm{c}}(1-B)^{-2}$, to obtain that

$$\Phi_{\mathrm{B}} \equiv \Phi_{\mathrm{B}}(\varepsilon_{\mathrm{v}}^{\mathrm{e}}, \varepsilon_{\mathrm{s}}^{\mathrm{e}}, B, \delta B) = \frac{\sqrt{E_{\mathrm{B}}(\varepsilon_{\mathrm{v}}^{\mathrm{e}}, \varepsilon_{\mathrm{s}}^{\mathrm{e}}) E_{\mathrm{c}}}}{1-B}\delta B \ge 0, \tag{3.8}$$

where $E_{\mathrm{B}}(\varepsilon_{\mathrm{v}}^{e}, \varepsilon_{\mathrm{s}}^{e})$ is given in equation (3.4).

Energy arguments were proposed in relation to grain size reduction ever since von Rittinger (1876) proposed his theory of comminution. In particular, the advantages of energy measures were highlighted in relation to the confined system of brittle particles. For example, Ueng & Chen (2000) associated the energy consumption from crushing linearly with the increase of surface area in order to interpret confined shear deformations, while Lade *et al.* (1996) quantified experimentally the constitutive relation between the input energy and the growth of various breakage measures during drained and undrained high-pressure triaxial shear tests.

With the current formulation, it is now possible to devise the explicit relations between the energy, breakage, stress and strain. In §4 the formulation is applied for constructing a student's model for sand; hence, it is probably appropriate to denote this special use of the theory as breakage soil mechanics. However, it is important to highlight that the theory is applicable to modelling any system of confined brittle granular assembly, not necessarily sand.

## 4. A simple student's model for sand

Guided by the discussion in §2, I present a model for sand based on breakage mechanics. The model is a student's model, adopting Muir Wood's (1990) hierarchical classification, in that it is made (deliberately) simple, yet is capable of teaching us several unexplained elements of sand behaviour. It requires only four physical parameters (shear and bulk moduli, $G$ and $K$, friction angle via $M$ and the critical breakage energy $E_{\mathrm{c}}$) in addition to knowledge of the initial sand characteristics of gsd (via $\vartheta$) and void ratio $e_0$. While being simple, it is sufficiently robust to allow defining a new hierarchy of models. For that, I start with the general model formulation and only later examine particular model case.

### (*a*) *General model formulation*

The increment of dissipation in equation (3.8) describes the energy consumption due to breakage only, i.e. directly from the specific surface area growth linked to the grading. It is important to realize that, in brittle granular materials, energy may be lost from an element from other sources that relate to rearrangement of particles (i.e. kinetic and potential energy losses from moving surrounding phonons and forming sound, and from dissipating energy by friction). These factors, in the current model, are notionally encapsulated in the increment of plastic dissipation. We assume a Coulomb type of increment of plastic dissipation

$$\Phi_{\mathrm{p}} \equiv Mp(\varepsilon_{\mathrm{v}}^{\mathrm{e}}, \varepsilon_{\mathrm{s}}^{\mathrm{e}}, B)|\delta\varepsilon_{\mathrm{s}}^{\mathrm{p}}| \geq 0, \tag{4.1}$$

where plastic dissipation from volumetric plastic strain is ignored; that is, in this direction, only the breakage increment dissipates energy. A fifth parameter can enable us to account for volumetric plastic strain (see Einav 2007*b*), but recall that the current aim is to end up with the simplest possible model. This compromise will allow us to construct a model that relies only on four parameters, but as we shall see, a model that clarifies many aspects of sand behaviour.

As discussed by Einav (2007*b*), the increments of breakage and plastic dissipation can be coupled to give the overall increment of dissipation $\Phi \equiv \sqrt{\Phi_{\mathrm{B}}^2 + \Phi_{\mathrm{p}}^2}$. Standard use of the degenerate special case of Legendre transformation allows us to replace the increments of the kinematics terms by their stress-like conjugate variables. This, combined with equation (3.4), enables us, after some elaboration, to derive the mixed stress-like space yield condition

$$y_{\mathrm{mix}} = \frac{E_{\mathrm{B}}(1-B)^2}{E_{\mathrm{c}}} + \left(\frac{q}{Mp}\right)^2 - 1 \leq 0. \tag{4.2}$$

Sand yielding is seen here as a function of the true stresses $q$ and $p$ and the breakage energy $E_{\mathrm{B}}$. The equation takes into account the active role of breakage in pure compression simply because when $q=0$ the criterion reduces back to equation (3.7). As the shear stress increases, the role of breakage is gradually becoming less dominant and frictional mechanisms take over. Mathematically, this is because more breakage energy would be needed. As $B$ tends to unity, the first term vanishes, and the above expression predicts the Mohr–Coulomb friction law given by $q=M\times p$. This is only the case when the grading is ultimate. $M$ does not change: this may be reasoned by arguing that the friction angle (for a given surface bearing area) is purely a function of mineralogy that does not change as well.

Using equations (3.2)–(3.4), it is generally possible to express the breakage energy as a function of 'unbroken effective stresses' $\bar{p} \equiv p/(1-\vartheta B)$ and $\bar{q} \equiv q/(1-\vartheta B)$, using $E_{\mathrm{B}} \equiv E_{\mathrm{B}}(\bar{p}, \bar{q})$. Given the latter relation, it is possible to rephrase the mixed stress space yield condition, and get a yield condition purely in true stress space. Depending on the free energy potential, the yield condition can be either explicitly or implicitly expressed in this space.

### (*b*) *Particular model case*

Let us examine the simplest case of the above formulation, assuming linear elasticity in the unbroken state,

$$\Psi = \frac{1}{2}(1-\vartheta B)\left(K{\varepsilon_{\mathrm{v}}}^2 + 3G(\varepsilon_{\mathrm{s}}^{\mathrm{e}})^2\right) = \frac{1}{2(1-\vartheta B)}\left(\frac{p^2}{K} + \frac{q^2}{3G}\right), \tag{4.3}$$

$$p = (1-\vartheta B)K\varepsilon_{\mathrm{v}}, \tag{4.4}$$

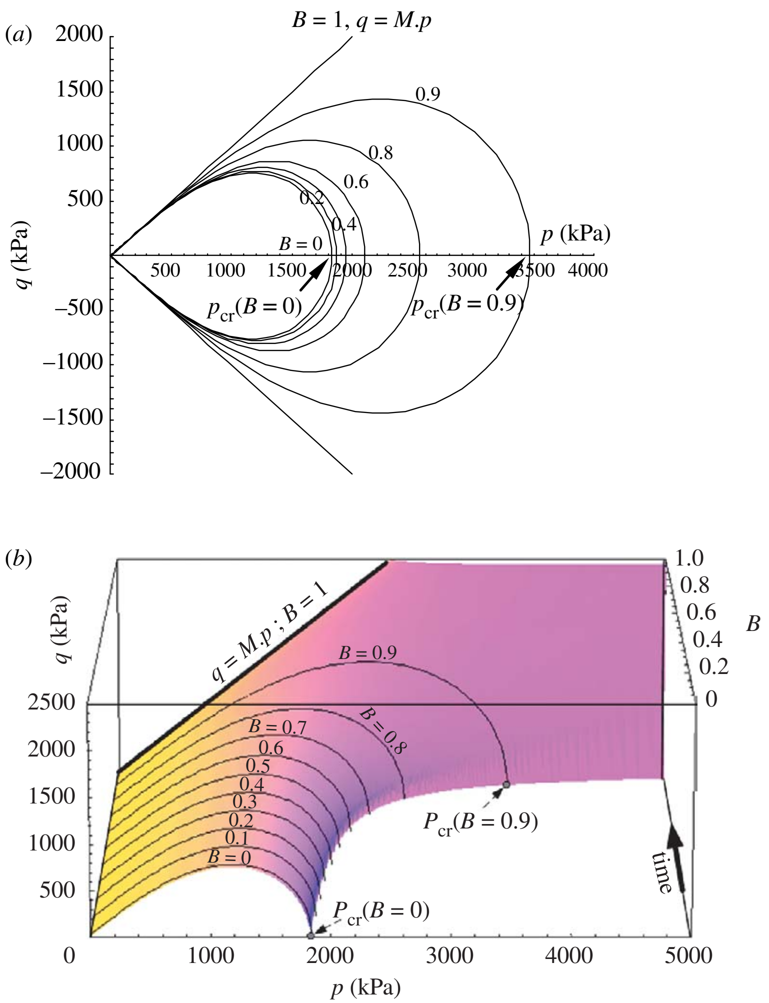


Figure 4. Two-dimensional and three-dimensional representations of the model's yield surface: (*a*) evolution of the yield surface projection on the $q-p$ stress-plane as a result of breakage growth and (*b*) the complete yield surface in $q$–$p$–$B$ space (drawn, for clarity, only for positive $q$). The figures correspond to the following four model's parameters: $K=30\,000$ kPa, $G=10\,000$ kPa, $E_c=50$ kPa and $M=1$. Additionally, the model material is well sorted, characterized by poorly distributed grading of $\vartheta=0.9$.

$$q = 3(1-\vartheta B)G\varepsilon_s^e \quad \text{and} \tag{4.5}$$

$$E_B = \frac{\vartheta}{1-\vartheta B}\Psi = \frac{\vartheta}{2(1-\vartheta B)^2}\left(\frac{p^2}{K}+\frac{q^2}{3G}\right). \tag{4.6}$$

Therefore, the yield condition in equation (4.2) can be replaced by the true stress space yield surface

$$y = \frac{\vartheta}{2E_c}\left(\frac{1-B}{1-\vartheta B}\right)^2\left(\frac{p^2}{K}+\frac{q^2}{3G}\right)+\left(\frac{q}{Mp}\right)^2 - 1 \le 0. \tag{4.7}$$

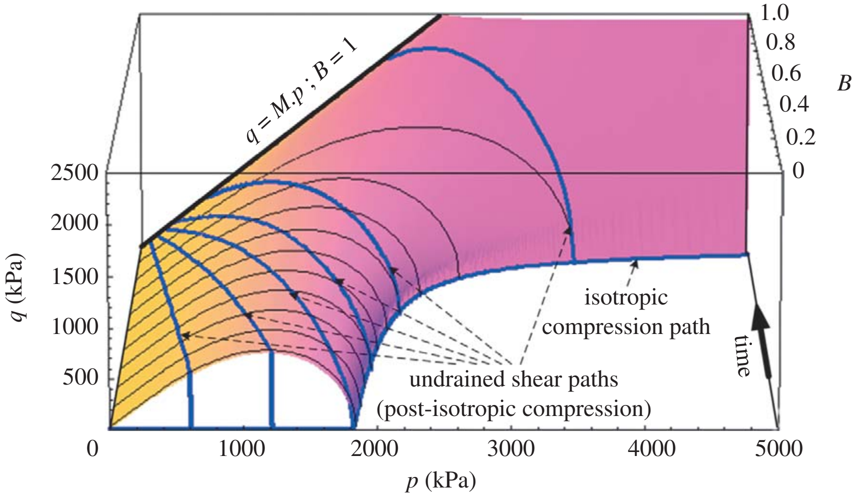


Figure 5. The stress-breakage paths (blue lines) in undrained shear tests, starting from various levels of consolidation, and superimposed on figure 4*b*.

The projection of this yield surface on the true stress $q$–$p$ plane is plotted as a function of $B$ in figure 4*a*. Initially, when $B=0$, the projection is the smallest, but as breakage grows the elastic region expands, eventually becoming the Mohr–Coulomb failure criterion. As the gsd evolves towards an ultimate fractal distribution, more energy is needed for further crushing. Breakage is therefore responsible for what is known as the isotropic hardening of the sand, in agreement with the conclusions by McDowell *et al.* (1996).

The complete yield surface may, in fact, be represented as a three-dimensional surface in $q$–$p$–$B$ space, as is shown in figure 4*b*. The surface envelops an elastic region that is always below the Mohr–Coulomb failure criterion, but that opens up as $B$ grows from zero to one, eventually merging with the $q=M\times p$ line. Prior to yielding or during unloading the breakage remains constant, and the stress point travels within the elastic region, along the constant $B$ planes. It is important to note that breakage is an entropic internal variable, because it cannot decrease but only grow with time. From a statistical mechanistic view point, this may be explained by referring to the statistical entropy of the gsd which always increases as $B$ increases, owing to the movement of the current gradation towards the more dispersed ultimate gradation. Radically novel in soil mechanics, the yield surface of sand is now seen connected to the elastic constants—but this, as I discuss in §5*b*, is supported by fracture mechanics.

The three-dimensional $q$–$p$–$B$ surface is a curved plane on which all of the stress path will travel once breakage is growing. For example, consider a series of undrained shear tests (constant volume tests), starting from a different level of consolidation. The simplicity of the current model easily enables us to derive analytical results, by combining equation (4.4) and (4.7) for constant volumetric strain $\varepsilon_v$. The results could then be superimposed on the three-dimensional surface, as is made in figure 5. Several tests are represented in this figure. Two of the tests correspond to moderately consolidated samples, with the shear phase of loading beginning from within the elastic region. Eventually, and after responding purely elastically to undrained shear, their stress paths hit the yield surface, inferring plastic shear strain and breakage. As of this moment $B$ grows towards unity and the stress-paths travel along the yield surface. Another test example corresponds to the case of precisely consolidating the sample to its initial compressive yield point $p = p_{cr}(B = 0)$. Subsequent shearing immediately introduces breakage. Finally, three other tests are consolidated to higher pressure levels, producing the breakage of 0.3, 0.6 and 0.9, even prior to shearing, but as shearing begins the stress paths travel along the $q$–$p$–$B$ surface.

All of the stress paths eventually end up residing on the $q = M \times p$ failure line (when $B = 1$). This feature means that if sufficiently strained all shear tests will end with a fully crushed ultimate grading. This pattern is intriguing because experience suggests that this would normally not be the case in loosely confined systems. While in strongly confined systems crushing becomes dominant, dilatancy governs the behaviour of loosely confined systems. In intermediate systems, crushing and dilatancy compete with the former tending to soften the material (as captured by the current model) and the latter hardening it. Given the simplicity of the model, I have decided not to introduce dilatancy at this stage, therefore the model is more efficient for moderately to high-pressured granular systems. Furthermore, the small effective elastic moduli of loose packings mean that such samples would need to deform quite considerably before actually reaching the ultimate grading. In these conditions, small-strain formulations of constitutive models cease to be adequate and higher-order finite-deformation forms of the model would have to be introduced.

## 5. What does the model teaches us?

The above model is simple, based on only four parameters with clear physical meaning ($K$, $G$, $M$ and $E_c$), consistent and physical, and enables us to define a hierarchy of new models. In view of §2 the question is: can the model teach us new things?

### (a) *Predicting evolving grading*

The presented model extends the notion of what a constitutive equation actually represents in soil mechanics. Traditionally, in soil mechanics, the definition is limited to describe the relations between stresses and strains. Breakage scales linearly, from zero to one, the position of the current gsd from the current and ultimate (fractal) gsd. Therefore, not only does the model connect between stresses and strains, but also the calculation of breakage allows us to predict how the gsd evolves with time. For example, Einav (2007*a*) examines this capability by comparing theoretical predictions to experimental results (figure 6).

### (b) *Yielding of sand: a link to fracture mechanics*

One of the cornerstones of material science is the Griffith theory (Griffith 1921) which presents an energy criterion for the fracture propagation of cracks in 'near-continuous' solids. Griffith suggested that the weakening of material by a crack could be treated as an equilibrium problem in which the reduction of strain energy, when the crack propagates, could be equated to the increase in surface

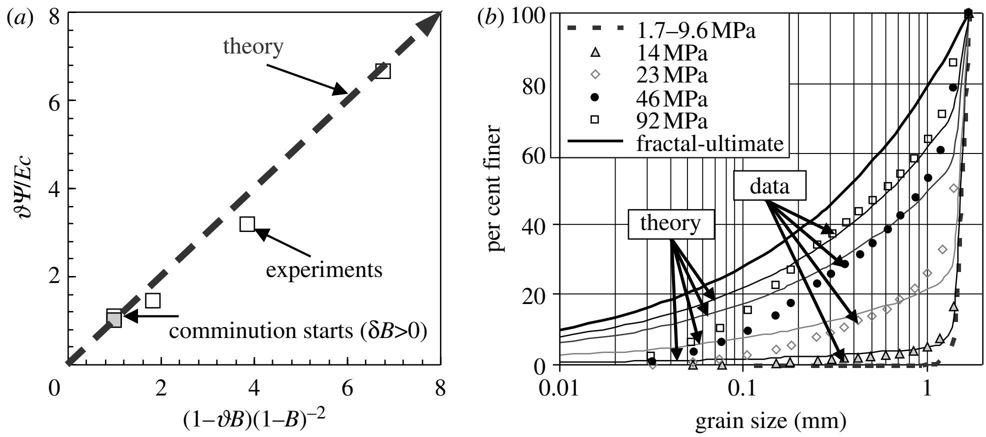


Figure 6. Theoretical (Einav 2007*c*) versus experimental (Nakata *et al.* 2001) relation between the vertical stress and breakage in one-dimensional compression of silica sand: (*a*) breakage hardening (dashed line is the arrow of time) and (*b*) the corresponding evolving gsd.

energy due to the increase in surface area. The result of the Griffith's energy balance criterion had enabled us to derive the celebrated formula to determine the critical tensile stress for causing a crack embedded in a plate to open,

$$\sigma_{\mathrm{cr}} = \sqrt{EG_{\mathrm{c}}/(\pi a)}.$$

This expression describes, in a very succinct way, the interrelation between three important aspects of the fracture process: (i) the material, as accounted by $G_{\mathrm{c}}$ and the Young's modulus $E$, (ii) the stress level $\sigma_{\mathrm{cr}}$, and (iii) the geometry of the crack using its length $2a$.

Analogously, the energy balance during confined comminution (i.e. the process of grain size reduction) is expressed in breakage mechanics via equation (3.6), where the loss of breakage energy is related to the loss in the available breakage energy for crushing particles. While fracture mechanics enable us to predict critical tensile stress for cracking, here we are interested in the critical pressure stress for comminution, obtained by using $B=0$ and $q=0$, and solving equation (4.7) for $p$,

$$p_{\mathrm{cr}} = \sqrt{2KE_{\mathrm{c}}/\vartheta}.$$

This relation bears a striking similarity to Griffith's equation for $\sigma_{\mathrm{cr}}$, describing succinctly the interrelation between three important aspects of the fracture process in brittle granular matter: (i) the material, using the critical breakage energy $E_{\mathrm{c}}$ (as an analogue to $G_{\mathrm{c}}$ in Griffith's expression) and Bulk modulus $K$ (the analogue to the Young's modulus E), (ii) the pressure level $p_{\mathrm{cr}}$ (the analogue to $\sigma_{\mathrm{cr}}$), and (iii) the geometry of the particles given by the normalized average grains surface area through $\vartheta$ (the analogue to the initial crack length $2a$).

The bulk modulus of granular material, $K$, is generally inversely proportional to the volume concentration (Walton 1987), such that we may write $K=(1/2)K_e(1+e)/e$ (with '$e$' being the void ratio and '$K_e$' denoting the bulk modulus when $e=1$). Therefore,

$$p_{\mathrm{cr}} = \sqrt{\frac{K_e(1+e)E_{\mathrm{c}}}{e\vartheta}}. \tag{5.1}$$

The last expression presents a factor that does not exist in Griffith's equation, considering the effect of the void ratio of the granular assembly. The result of this addition is logical, suggesting that agglomerates would break earlier if initially looser; this is in agreement with experimental results (figure 2*b*). This formula further reveals the effect of the gsd on the critical pressure for producing comminution. As the initial gradation is closer to the ultimate gradation, $\vartheta$ is smaller and $p_{\rm cr}$ increases.

Einav (2007*c*) further investigates the connection of the above analysis to fracture mechanics, by relating breakage to specific surface area, and examines the effect of assuming nonlinear Hertzian elastic model, instead of using linear elasticity; the basic idea is the same. Therefore, one of the hierarchical extensions to the currently presented model could simply be executed by replacing the linear compression–elasticity with nonlinear Hertzian elasticity. This would slightly distort figure 6 (because the dependence of $E_{\rm B}$ on $p$ would change), but will not change the conclusions.

### (*c*) *CSL: the dependence on the initial grading and pre-consolidation pressure*

Finally, it is possible to examine how the model 'looks at' the concept of the CSL, i.e. the line on which sheared samples would no longer change their volume. As demonstrated in figure 1, by Finno & Rechenmacher (2003), this line is no longer unique, and is affected by the initial grading and pre-consolidation pressure. It is now well established that this result is the outcome of breaking the particles and constantly changing the material (fabric). In the current model, for any given value of breakage, the pressure is linearly proportional to the volumetric strain through $p=(1-\vartheta B)K\varepsilon_{\rm v}$. The volumetric strain is, by itself, a function of the void ratio, e.g. using the simplifying equation $\varepsilon_{\rm v}\approx\log(1+e_{\rm max})-\log(1+e)$. The CSL of pre-compressed samples that did not experience any crushing during the compression stage would be given by

$$p_{\rm CSL}(e_{\rm CSL})=(1-\vartheta)K(\log(1+e_{\rm max})-\log(1+e_{\rm CSL})), \tag{5.2}$$

where on the CSL the breakage is fully exhausted (i.e. $B=1$), and the pressure and void ratio are denoted as $p_{\rm CSL}$ and $e_{\rm CSL}$. This implies the first type of the non-uniqueness of the CSLs, where the effect of the initial grading is introduced through the presence of $\vartheta$. As $\vartheta$ is larger, i.e. as the initial gsd is further away from the ultimate distribution, the potential of the virgin material to undergo fabric changes is higher, hence pulling down the CSL towards lower values of critical state pressure. The current analysis further shows that the maximum void ratio affects the position of the CSL as well, and that $p_{\rm CSL}$ gets smaller as the critical void ratio $e_{\rm CSL}$ becomes closer to $e_{\rm max}$.

However, note that while pre-consolidation may infer crushing and fabric changes, the above equation is characteristic to the current model. In reality $e_{\rm max}$ depends on the grading, i.e. on $B$. This suggests that pre-consolidation breakage would permanently change $e_{\rm max}$, giving a family of parallel lines with the level depending on the inferred breakage during the pre-consolidation stage $B_{\rm NC}$.

Another possible way to illustrate this is using the concept of the unbroken effective pressure $\bar{p}\equiv p/(1-\vartheta B_{\rm NC})$. Alternatively, we can define the 'unbroken effective strain' as $\bar{\varepsilon}_{\rm v}\equiv\varepsilon_{\rm v}(1-\vartheta B_{\rm NC})$. Therefore, to be able to look at broken consolidated samples as unbroken consolidated samples, we can multiply equation (5.2) by $(1-\vartheta B_{\mathrm{NC}})$

$$p_{\mathrm{CSL}}(e_{\mathrm{CSL}}) = (1-\vartheta)(1-\vartheta B_{\mathrm{NC}})K(\log(1+e_{\max})-\log(1+e_{\mathrm{CSL}})). \tag{5.3}$$

This equation reverts back to equation (5.2) when $B_{\mathrm{NC}}=0$. As the pre-consolidation stage produces larger breakage of $B_{\mathrm{NC}}$ during the pre-consolidation stage, $p_{\mathrm{CSL}}$ becomes smaller for the same void ratio, in agreement with figure 1*b*,*c*.

This analysis can be extended in various ways. For example, nonlinear elasticity and the relation $K=(1/2)K_e(1+e)/e$ would automatically change equation (5.2), distorting the shape of the CSLs, but not the conclusions. The model is thus wide in its scope of illustrating the physical reasons behind the non-uniqueness of the CSL in sand, even in its most simple form. Radically in soil mechanics, the model looks at the concept of CSL as an emerging property, rather than imposing this equation. In its hierarchical form, the model therefore facilitates a pioneering vehicle for studying theoretically the effect of various parameters on the non-uniqueness of CSL, shedding new light on previous experimental and numerical studies (Yamamuro & Lade 1998; Finno & Rechenmacher 2003; Cheng *et al.* 2005).

## 6. Leaving the past and living the future

We have tried for many years to approach the continuum modelling of sand using the theory of CSSM, but unsuccessfully by adding ad hoc fitting parameters to clay models. However, we must acknowledge that the micro-structure and mechanics of sand are different than those of clay, which is why continuum modelling must take a different path.

This article proposes an alternative approach to the modelling of the critical effects of sand, using breakage mechanics rather than CSSM. The theory is based on the gsd and its evolution, and this was integrated to construct a simple student's constitutive model for sand. With only four physical parameters, the model is capable to teach us about several unexplained elements of behaviour, specifically the dependence of yielding and failure on the initial gsd and void ratio. The model is capable of predicting how the gsd evolves, in relation to changing stresses and strains, extending the applicability of geotechnical constitutive models to problems that were until now beyond reach.

It is important to note that here 'sand' can actually refer to any other confined system of brittle granular material, and the theory of breakage mechanics is applicable to not only geotechnical modelling, but also many other disciplines that deal with confined comminution, including geophysics, geology, powder technology, mineral processing, agriculture, the food industry and pharmaceutics. Future trends of research consistently demand moving towards cross-disciplinary challenges, which require technology and knowledge transfer. To maximize this, theories are required, which are physically based on the characteristics of the materials, such that moving from modelling sand–sand, to modelling say pasta–sand or wheat–sand, would only mean changing physical parameters. It is my personal view that breakage mechanics offer a very promising research direction for achieving that.

I have enjoyed receiving some very helpful tips from my colleagues at the University of Sydney; particularly from Prof. David Airey and Dr Gianluca Ranzi. In regards to this work, I would then like to thank great stimulus from some external collaborators (in alphabetical order): Prof. Andrew Chan, Prof. Antoinette Tordesillas, Prof. Julio Valdes and Prof. Ioannis Vardoulakis. Finally, the support of the Australian Research Council, through the ARC Discovery grant scheme (DP0774006), is gratefully acknowledged.

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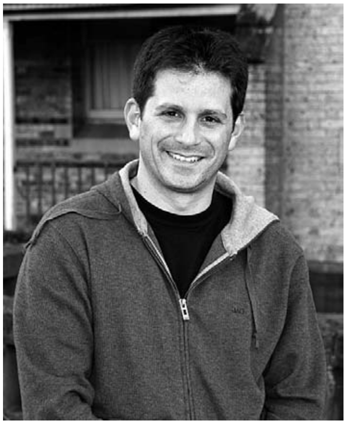
# AUTHOR PROFILE

## Itai Einav



Born in Jerusalem (Israel), Itai Einav studied civil engineering at the Technion, Haifa, where he graduated with a BSc in 1998 and a PhD in 2002. He then moved to the University of Western Australia, COFS, Perth, to become a Research Associate, then Research Fellow, and then Senior Research Fellow. During this time, he received an Australian Postdoctoral Fellowship from the Australian Research Council (ARC) and was awarded an MTS Visiting Professorship from the University of Minnesota. In 2005, after 3 years in Perth, he moved to the other side of the continent to become a Senior Lecturer at the University of Sydney, in the School of Civil Engineering. Soon, he discovered an attraction to breakage and received another grant from the ARC to study this topic. Aged 35, his main scientific interests include theoretical granular mechanics and physics, applied (onshore and offshore) geotechnics and characterization of random materials. Recreations include travelling (above and below water level), kayaking and surviving the gym.