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Fracture propagation in brittle granular matter

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It is nearly a century since Alan Arnold Griffith developed his energy criterion for the fracture propagation of cracks in 'near-continuous' solids. Needless to say that his celebrated work has revolutionized the world of material science. In a very succinct way, Griffith connected between three important aspects of the fracture process: (i) the material, (ii) the stress level, and (iii) the geometry of the crack. Nothing similar was developed for brittle granular matter, although in these materials fracture propagates in the sense of comminution. Recently, I have developed an energy theory, called breakage mechanics, based on the concept of breakage. However, the analogy between the mechanics of breakage and fracture is missing. Here I establish this relation using energy principles and derive a critical comminution pressure for brittle granular materials. This critical pressure is surprisingly complementary to Griffith's critical tensile stress for near-continuous materials. This step enables for the first time to apply the principles of fracture mechanics to all disciplines dealing with confined particles comminution such as geophysics, geology, geotechnical engineering, mineral processing, agriculture and food industry, pharmaceutics and powder technology.

Keywords: granular matter; comminution; fracture mechanics; breakage mechanics; thermomechanics

1. Introduction

Granular materials are among nature's most versatile mechanical systems. They behave as a gas or a fluid if their grains are unconfined, but as a solid granular agglomerate if sufficiently confined and the jamming of force chains occurs (e.g. Aharonov & Sparks 1999; Corwin *et al.* 2005). When the confining pressure is sufficiently large grain crushing may commence. Energy principles have been applied to study size reduction of particles ever since von Rittinger (1867) and Kick (1883) developed their theory of comminution. Rittinger suggested that the new surface area produced is proportional to the energy consumed, while Kick proposed that relative reduction occurs irrespective of the original size. Although their work is the starting point for many solutions in mineral processing, much controversy arose about their hypotheses as other workers produced results to satisfy either one or the other (Lynch 1977). The prevailing problem is that they do not connect the factors which govern energy consumption, i.e. the boundary

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conditions of the problem and the rheology of the material. This fact has been known for many years to geotechnical engineers, developing constitutive relations that connect between stresses and strains. However, the problem with most of these relations is that they are purely phenomenological. Only minimal attention was paid to develop models that relate to the evolution laws at the physical microscopic level of crushing (e.g. Papamichos *et al.* 1993; McDowell *et al.* 1996).

One of the cornerstones of material science (e.g. Gordon 1988) is the Griffith theory (Griffith 1921) which presents an energy criterion for the fracture propagation of cracks in 'near-continuous' solids. Griffith suggested that the weakening of material by a crack could be treated as an equilibrium problem in which the reduction of strain energy, when the crack propagates, could be equated to the increase in surface energy due to the increase in surface area. Figure 1*a* presents the schematic of the problem Griffith addressed, where a plate with an initial elliptical hole of the length $2a$ is tensioned, so that after the fracture propagates the length of the elliptical hole becomes $2(a+\delta a)$. Griffith found that the critical stress to cause a crack to extend is $\sigma_{\mathrm{cr}}=(2E\gamma/\pi a)^{1/2}$, where E is Young's modulus and γ is the surface energy. Griffith's original work dealt with very brittle materials. To account for the material ductility, Irwin (1957) suggested that catastrophic fracture occurs when the released strain energy is absorbed by energy dissipation due to plastic flow in the material near the crack tip, and denoted the critical strain energy release rate by the parameter G_{c}; the Griffith equation can then be rewritten as follows:

$$\sigma_{\mathrm{cr}} = \sqrt{\frac{EG_{\mathrm{c}}}{\pi a}}. \tag{1.1}$$

This expression describes, in a very succinct way, the interrelation between three important aspects of the fracture process: (i) the material, as accounted by G_{c} and Young's modulus E, (ii) the stress level σ_{cr} and (iii) the geometry of the crack using its length $2a$.

Griffith's analysis, however, was concerned with near-continuous brittle materials. Nothing similar exists to describe brittle granular matter—a particulate system that can no longer be defined as near continuous. The elementary problem we address is shown schematically in figure 1*b*, where isotropic compression is identified as the fundamental problem to be solved. Recently, I have developed an energy theory, called breakage mechanics (Einav 2007*a*), based on the concept of breakage. However, the analogy between the mechanics of breakage and fracture is missing. Here I develop this relation and concisely derive a formula for the critical comminution pressure in confined brittle granular matter (i.e. a formula for the pressure at the onset of significant grain crushing). This critical pressure is astonishingly complementary to Griffith's critical fracture tensile stress for near-continuous materials.

2. Breakage and self-organized criticality

The elementary problem addressed in this paper is shown schematically in figure 1*b*. We start by considering a confined configuration under isotropic pressure conditions. A minimal reference pressure p_{r} must be applied before the jamming of force chains occurs and the granular material may be considered as a

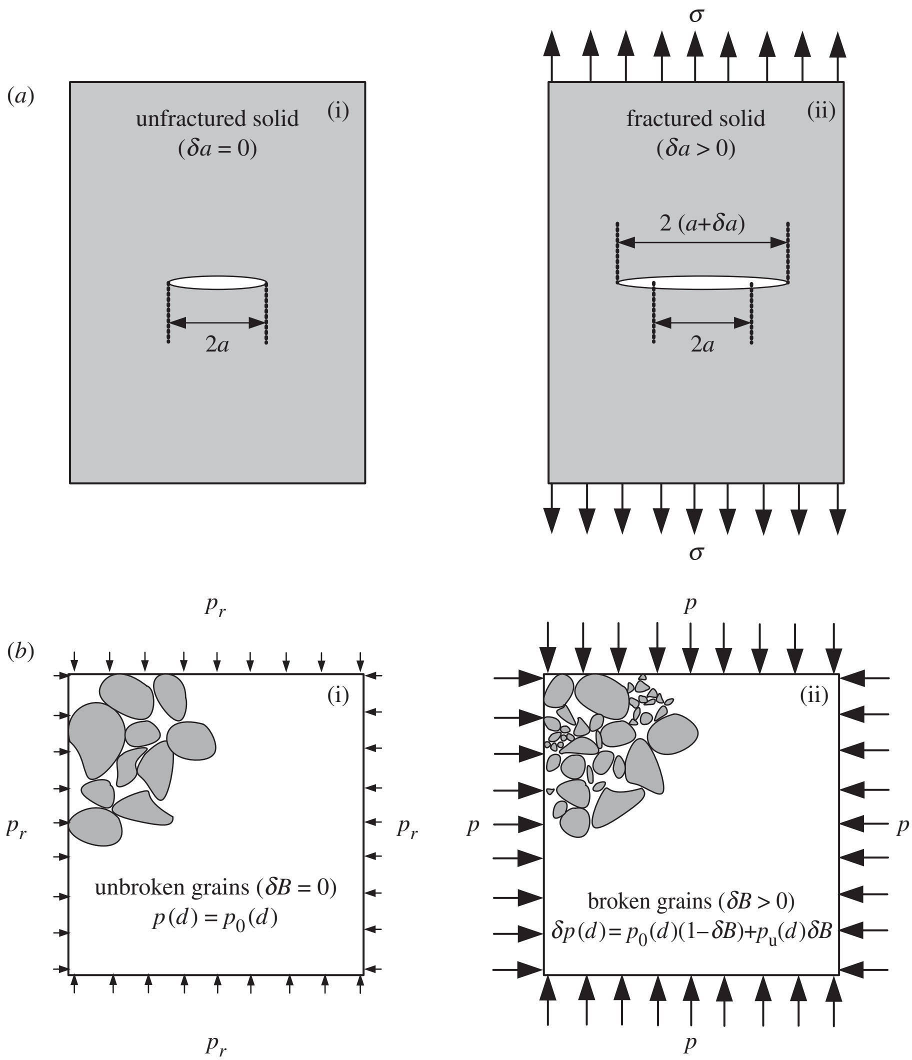

Figure 1. Fracture propagation. (a) The schematic of Griffith (1921) problem of tensioned 'near-continuous' solid plates with an initial elliptical hole of length $2a$: (i) before and (ii) after. (b) The problem of compressed granular agglomerates ($p(d)$ defines the grain size distribution as a probability density function; not to be confused with p which stands for the pressure): (i) before and (ii) after.

solid granular agglomerate, distinguished from granular fluids and granular gasses (e.g. Aharonov & Sparks 1999). Although not referring to jamming as such, in soil mechanics this pressure is tacitly assumed at the order of 1 kPa; a much smaller pressure before crushing starts. Under these conditions, we may apply the principles of continuum mechanics. It is important to note that the fracture in granular materials is linked to the evolving statistics of the grain size distribution. Before some critical pressure is reached, granular systems continue to be jammed, without any changes to the grain size distribution. After the commencement of grain crushing, the entire grain size distribution is changed from a statistical view point. At this stage, the jammed configuration of the force chains gets unjammed again, and the fragments can self-organize. Crushing is therefore playing an active role under isotropic compression, without which no dissipation occurs. Therefore, for the study of critical comminution pressure, one can neglect the other sources of energy consumption and this is exactly why we chose isotropic compression as the fundamental problem to solve. Plastic dissipation does exist, and can be introduced by adopting the coupling breakage–plasticity formulation of Einav (2007*b*), but the derived formula of the critical comminution pressure would remain as in the current analysis. There is a relation between the way the grain size distribution evolves and how the grains organize. This connection may be efficiently described using the concept of the measurable quantity of breakage B (Einav 2007*a*). Similar definitions for breakage were given by Hardin (1985) and Wood (2006).

The emergence of the critical behaviour of many systems does not depend on finely tuning the details of the system (Bak *et al.* 1987). Evolving systems that have a critical point as an attractor (a state to which the system evolves after a long enough time) present self-organized criticality as a property. It is possible to view confined comminution as one of such processes. In detail, comminution of brittle granular systems tends to minimize the occurrence of any disproportionate shear stress that individual particles might carry (McDowell *et al.* 1996). As bigger particles are cushioned more, they carry larger normal contact forces, but this is counterbalanced by the smaller particles that statistically attract less contacts, which lead to higher stress concentration within them. Eventually, as comminution progresses and the unbalanced shear stresses fall bellow a critical level to cause further crushing, an apollonian-like organization tends to form a hint to fractal nature and self-similarity (Sammis *et al.* 1987). The fractal grain size distribution may therefore be seen as the strange attractor to confined comminution. Although it is important to note that under certain pathological conditions (e.g. crushing of initially bi-sized mixtures), a pure fractal distribution would not emerge, an alternative ultimate grain size distribution will evolve, following a different path of organization. In many naturally occurring conditions, where pure fractal scale tends to emerge, the grain size distribution becomes a power law, inversely proportional to the grain sizes. The ultimate cumulative grain size distribution by mass $F_{\mathrm{u}}(d)$ also turns out to be a power law (Turcotte 1986), with grain sizes ranging below the largest grain size in the original and ultimate packings d_{M}. Consideration of fracture mechanics at the level of the individual grains shows that the grain sizes should be bounded from below by a minimal grain size d_{m} (Kendall 1978). Differentiating the cumulative distribution gives the probability density function (pdf) of the grain sizes $p_{\mathrm{u}}(d)$. Starting with an initial distribution $p_0(d)$ (or $F_0(d)$), the current distribution is being attracted to the ultimate distribution. We may define the current state by a measurable internal state variable B, which we call the 'breakage'. This property denotes the relation between the initial, current and ultimate (which could be fractal or not) cumulative grain size distributions as shown in figure 2*a*.

After viewing the ultimate distribution as an attractor, we can physically interpret breakage as a measure of the criticality proximity, with a value bounded between 0 and 1, $1 \geq B \geq 0$. When B approaches unity, the system tends to criticality. We take the strong hypothesis that this measure is fractionally

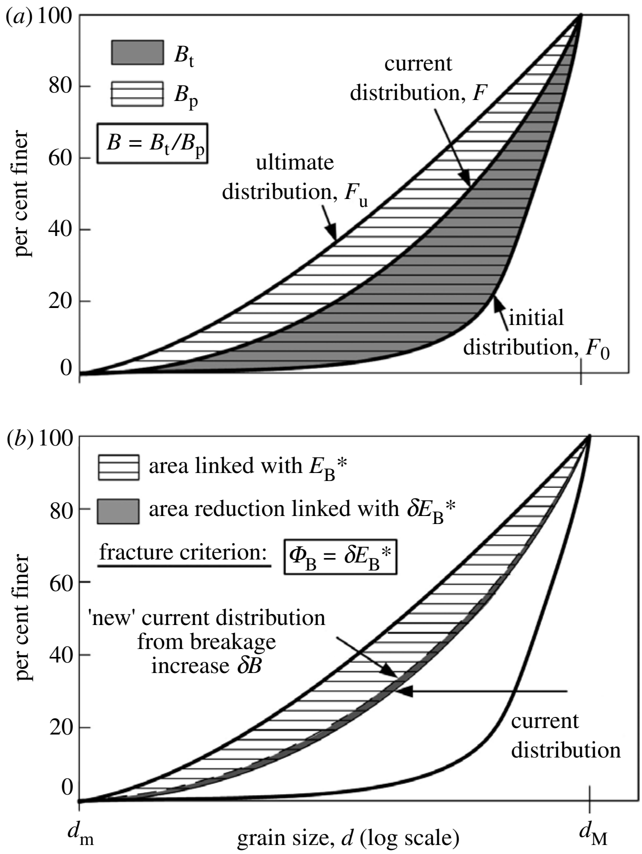

Figure 2. (*a*) The measurable definition of breakage (Einav 2007*a*). (*b*) The breakage propagation criterion for granular materials (Einav 2007*a*). The incremental breakage dissipation is equal to the incremental change in the residual breakage energy.

independent and write as follows:

$$p(d) = p_0(d)(1-B) + p_u(d)B. \tag{2.1}$$

The fractional identity hypothesis enforces the different points along the current cumulative distribution to scale similarly to the corresponding points along the initial and ultimate distributions, but the usefulness of this assumption was proven (Einav 2007*a*).

In its incremental form, equation (2.1) becomes

$$\delta B = \delta p(d)/(p_u(d) - p_0(d)). \tag{2.2}$$

Therefore, the evolution of breakage, δB, effectively spans the entire variation of the grain size distribution $\delta p(d)$.

3. Breakage energy

Let $x(d)$ be an average quantity for all grains with a diameter that lies within the fraction of sizes $(d-\mathrm{d}d/2)\leq d\leq(d+\mathrm{d}d/2)$. We use the conventional notation for the statistical average of this variate within the entire assembly (i.e. a representative quantity for all fractions), using the pdf of the grain sizes

$$X \equiv \langle x \rangle = \int_{d_{\rm m}}^{d_{\rm M}} p(d)x(d)\mathrm{d}d, \tag{3.1}$$

so that using equation (2.1), we have

$$X \equiv \langle x \rangle = (1-B)\langle x \rangle_0 + B\langle x \rangle_{\rm u}, \tag{3.2}$$

where $\langle x \rangle_0$ and $\langle x \rangle_{\rm u}$ are the statistical averages of $x(d)$ using the initial $p_0(d)$ and ultimate $p_{\rm u}(d)$ grain size distributions. By accepting the additivity property of energy, we may express the statistical average of the stored energy of the entire agglomerate by

$$\Psi \equiv \langle \psi \rangle = (1-B)\langle \psi \rangle_0 + B\langle \psi \rangle_{\rm u}, \tag{3.3}$$

where $\psi \equiv \psi(d)$ designates the energy density function of a given fraction size d, an average quantity for all the particles in the fraction itself.

Within the rigorous confinement of thermodynamics (Einav 2007*a*), we define the 'breakage energy' and 'residual breakage energy' by

$$E_{\rm B} = -\partial\Psi/\partial B = \langle \psi \rangle_0 - \langle \psi \rangle_{\rm u} \quad \text{and} \tag{3.4}$$

$$E_{\rm B}^* = E_{\rm B}(1-B) = (1-B)(\langle \psi \rangle_0 - \langle \psi \rangle_{\rm u}) = \langle \psi \rangle - \langle \psi \rangle_{\rm u}. \tag{3.5}$$

We see that the breakage energy $E_{\rm B}$ relates to the difference between the macroscopic statistical averages of the stored energy using the initial and the ultimate distributions. Before any crushing occurs, this energy difference denotes the potential of the granular system to be pushed towards criticality. However, it is important to define this potential at any stage of the loading process, i.e. to assess the tendency of the material to reach criticality after crushing has already occurred. Therefore, what we actually need is the difference between the current and ultimate energy averages. This is precisely the reason for defining the residual breakage energy $E_{\rm B}^*$.

Combining equations (3.3) and (3.4) gives the following relation:

$$\Psi = \langle \psi \rangle_0 - E_{\rm B}B. \tag{3.6}$$

4. Energy balance in comminution

Griffith (1921) assumed that growth of a crack in near-continuous solids requires creation of surface energy, which is supplied by the loss of strain energy accompanying the relaxation of local stresses as the crack advances. Fracture occurs when the loss of strain energy is sufficient to provide the increase in surface energy. We need a similar criterion for granular materials, but the problem is that their topology is much richer than the cracked plate problem. For our advantage, however, the increase in surface area can be analysed for a statistical assembly. Figure 2*b* denotes the breakage area relating to the residual breakage energy $E_{\rm B}^*$, using equation (3.4). Recall that this energy denotes the remaining ability of the system to break particles towards reaching criticality. As figure 2*b* shows, a change in breakage carries changes to the residual breakage area, highlighted by the grey-crescent area in the figure. This incremental area corresponds only to fragments that comminuted at that particular instance. Therefore, the energy that the material consumes at a given moment, or the incremental change in breakage dissipation Φ_{B}, must correspond to the loss in residual breakage energy (Einav 2007*a*)

$$\Phi_{\mathrm{B}} = \delta E_{\mathrm{B}}^{*} \geq 0, \tag{4.1}$$

where the inequality denotes 'propagation' and the equality is concerned with unloading or the non-comminuting elastic state. This equation is analogous to Griffith's energy equilibrium. The fragmentation creates new surface energy, which is supplied by the loss of strain energy accompanying the relaxation of local internal contact forces as the grains get smaller and unjamming occurs. We note that $\delta E_{\mathrm{B}}^{*} = (1-B)\delta E_{\mathrm{B}} - E_{\mathrm{B}}\delta B$. Within the confines of rate-independent processes, the dissipation is assumed as first order in the rate of the internal variables (here, the breakage), i.e. $\Phi_{\mathrm{B}} = (\partial \Phi_{\mathrm{B}}/\partial \delta B)\delta B$ and that $\partial \Phi_{\mathrm{B}}/\partial \delta B$ be the dissipative version of the energy breakage E_{B}. Therefore, we have $(1-B)\delta E_{\mathrm{B}} - 2E_{\mathrm{B}}\delta B = 0$. This is the evolution equation of breakage during comminution. Integrating this equation gives the breakage yielding condition

$$y_{\mathrm{B}} = E_{\mathrm{B}}(1-B)^2 - E_{\mathrm{c}} \leq 0, \tag{4.2}$$

where E_{c} is a new constant of integration, denoting the critical breakage energy E_{B} needed to start causing particles to crush (i.e. when B is still 0). The status of this constant is similar to Griffith's energy constant G_{c}, but here E_{c} is taken as a statistical measure of the granular agglomerate. This constant is believed to relate to microscopic material properties (the particle mineralogy, local particle defects and their angularity) and not on the grain size distribution and the macroscopic void ratio. When particles break, breakage yielding gives $E_{\mathrm{B}} = E_{\mathrm{c}}(1-B)^{-2}$, and energy is dissipated. Therefore, the breakage dissipation may be expressed explicitly by

$$\Phi_{\mathrm{B}} = E_{\mathrm{B}}\delta B = E_{\mathrm{c}}(1-B)^{-2}\delta B. \tag{4.3}$$

We would like to examine the relation of the above breakage growth criterion in the sense of fracture mechanics, i.e. in terms of surface area growth. For that purpose, the problem is simplified for the analysis of a compressed system of perfectly incompressible elastic-brittle spheres. Since the overall mass of solids in granular agglomerates has to remain constant irrespective of the reduction in the fragment sizes (and the total increase in their number), it is the average-specific surface area (i.e. the average of the particles' surface area divided by their volume) that we have to consider. Since for spheres the specific surface area of a particle with diameter d is $s(d) = 6/d$,

$$S \equiv \langle s \rangle = (1-B)S_0 + BS_{\mathrm{u}}, \tag{4.4}$$

where the initial and ultimate specific areas are denoted by $S_0 = 6\langle d^{-1}\rangle_0$ and $S_{\mathrm{u}} = 6\langle d^{-1}\rangle_{\mathrm{u}}$, respectively. These parameter constants relate linearly to the inverse of the initial and ultimate mean harmonic grain sizes. This equation can be solved for B and its incremental change δB, giving

$$B = (S - S_0)/(S_{\mathrm{u}} - S_0) \quad \text{and} \tag{4.5}$$

$$\delta B = \delta S/(S_{\mathrm{u}} - S_0). \tag{4.6}$$

Therefore, a duality is established between the breakage mechanics formulation and the specific surface area approach, as in fracture mechanics,

$$\Psi = \frac{S_u - S}{S_u - S_0}\langle\psi\rangle_0 + \frac{S - S_0}{S_u - S_0}\langle\psi\rangle_u \quad \text{and} \tag{4.7}$$

$$\Phi_B = E_c \frac{S_u - S_0}{(S_u - S)^2}\delta S. \tag{4.8}$$

Since the specific surface area of particles get larger as their fragments get smaller, we have $S_u \geq S \geq S_0$. Initially, the mean specific surface area in the assembly is S_0, hence $\Psi = \langle\psi\rangle_0$, but as $S \to S_u$, $\Psi \to \langle\psi\rangle_u$. Furthermore, the analysis suggests that more energy is needed for breaking particles as $S \to S_u$, since the denominator in Φ_B approaches 0. Finally, we may rephrase the breakage condition $y_B = 0$, as

$$\frac{E_B}{E_c} = \left(\frac{S_u - S_0}{S_u - S}\right)^2. \tag{4.9}$$

To establish the analogy between breakage dissipation and fracture dissipation, it is possible to define the specific surface area energy, as the thermodynamics conjugate to the mean specific surface area S,

$$E_S = -\partial\Psi/\partial S = E_B/(S_u - S_0). \tag{4.10}$$

Therefore,

$$\frac{E_S}{E_c} = \frac{S_u - S_0}{(S_u - S)^2}. \tag{4.11}$$

Combining the last relation and equation (4.8) gives the relation

$$\Phi_B = \Phi_S = E_S\delta S. \tag{4.12}$$

This relation resembles the increment of non-negative entropy production, as defined by Rice (1978) in his work on fracture mechanics. The property E_S is equivalent to specific surface energy γ as in Griffith's analysis. It shows that the breakage dissipation essentially denotes the energy consumption from the creation of new (mean specific) surface area, as in fracture mechanics.

5. Critical comminution pressure

In granular systems, bigger particles tend to attract more contact points due to their larger surface area. Therefore, the energy that is stored on average in the individual particles may be expected to scale linearly with the surface area, in a universal manner and independent of the grain size distribution (neglecting the pathological near-crystalline distributions). Figure 3 examines this hypothesis based on three different distinct element method simulations, each using a different grain size distribution. The first simulation presents the result from a system of two-dimensional grains with a (discrete) fractal distribution of sizes $N_d \propto d^{-1.5}$ for five distinct sizes. The other two simulations are based on (continuous) uniform distributions of grain sizes (by number) with the ratio of biggest-to-smallest particle being 1.5 and 20. These distributions were specifically selected to be as versatile in character as possible, examining the above scaling

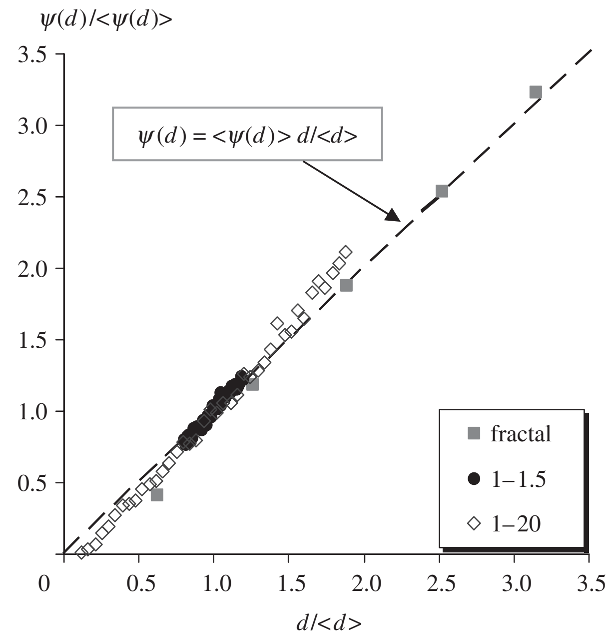

Figure 3. The universality of the energy stored in the different fraction sizes, shown to scale linearly with the particle surface area.

hypothesis for granular materials ranging from being poorly graded, well-graded and ultimately fractal distributions. Albeit minor deviations, attributed mainly to the numerical overlaps between the particles, it is important to note that the average stored energy in the various fractions $\psi(d)$ scales linearly with the grain surface area (in the system of discs, taken as πd), almost perfectly universally and independent of the imposed grain size distribution.

In a system of discs, the surface area is proportional to the grain sizes, but in a system of spheres, the surface area is proportional to the diameter squared, therefore

$$\psi(d) = \langle \psi(d) \rangle d^2/\langle d^2 \rangle = \Psi d^2/\langle d^2 \rangle. \tag{5.1}$$

Since in figure 3 it was argued numerically that this relation does not depend on the specific grain size distribution, we can use either the initial or ultimate grain size distributions, and write

$$\langle \psi(d) \rangle_0/\langle d^2 \rangle_0 = \langle \psi(d) \rangle_{\mathrm{u}}/\langle d^2 \rangle_{\mathrm{u}}. \tag{5.2}$$

Combining the equations (5.2) and (3.4) shows that in the system of spherical particles the breakage energy becomes

$$E_{\mathrm{B}} = \vartheta \langle \psi \rangle_0, \tag{5.3}$$

so with equation (3.6), we have

$$E_{\mathrm{B}} = \vartheta \Psi/(1-\vartheta B), \tag{5.4}$$

where the parameter

$$\vartheta = 1-\langle d^2 \rangle_{\mathrm{u}}/\langle d^2 \rangle_0, \tag{5.5}$$

measures how far the initial grain size distribution is from the ultimate distribution, in terms of the averaged surface area (or alternatively the second-order moments of the distributions). In other words, this property depends on geometrical scales, measuring the initial proximity to criticality. As specified by equation (4.2), the yielding in a compressed system of brittle grains is given by $E_{\mathrm{B}}(1-B)^2 = E_{\mathrm{c}}$. Therefore, using equation (5.4), we write the critical energy required to be stored in the agglomerate for initiating comminution

$$\Psi_{\mathrm{cr}} = E_{\mathrm{c}} \frac{(1-\vartheta B)}{\vartheta(1-B)^2}. \tag{5.6}$$

Before first crushing, we have $\Psi_{\mathrm{cr}} = E_{\mathrm{c}}/\vartheta$. In an isotropically compressed linear elastic system, the bulk modulus is pressure independent and $\Psi = p^2/2K$, giving

$$p_{\mathrm{cr}} = \sqrt{\frac{2KE_{\mathrm{c}}}{\vartheta}}, \tag{5.7}$$

where p_{cr} is the critical comminution pressure; K is the bulk modulus; and E_{c} is the critical breakage energy constant.

This relation bears striking similarity to Griffith's equation (1.1) for near-continuous solids. The above expression describes, in a very succinct way, the interrelation between three important aspects of the fracture process in brittle granular matter: (i) the material, as evidenced in the critical breakage energy E_{c} (as an analogue for G_{c} in Griffith's expression) and bulk modulus K (as an analogue for Young's modulus E), (ii) the pressure level p_{cr} (as an analogue for σ_{cr}), and (iii) the geometry of the particles given by the normalized average grain surface area through ϑ (as an analogue to the initial crack length $2a$).

The initial average surface area $4\pi\langle d^2\rangle_0$ is always larger than the ultimate average surface area $4\pi\langle d^2\rangle_{\mathrm{u}}$, suggesting that ϑ is less than unity, but greater than 0. As closer the initial grain size distribution is to the ultimate distribution, ϑ becomes closer to zero, and the critical pressure gets larger.

It is well known, however, that the bulk modulus of granular matter is in fact a power function of the pressure, given by $K = p_{\mathrm{r}} K^* \ (p/p_{\mathrm{r}})^m$, where K^* is a dimensionless constant and m is the Herzian constant. This suggests that $\Psi = p_{\mathrm{r}}(p/p_{\mathrm{r}})^{2-m}/(1-m)(2-m)K^*$. In a system of randomly compacted spheres, the Herzian constant becomes $m = 1/2$ (Walton 1987). Hence, to account for the pressure dependency of granular materials, the critical pressure for the start of confined comminution in a system of brittle spheres is given by

$$p_{\mathrm{cr}} = p_{\mathrm{r}} \sqrt[3/2]{\frac{3K^* E_{\mathrm{c}}}{4\vartheta p_{\mathrm{r}}}}. \tag{5.8}$$

Furthermore, the effective elastic moduli of granular materials are inversely proportional to the porosity (e.g. Walton 1987; Pestana & Whittle 1995); in terms of the macroscopic void ratio e, we may write the dimensionless constant of the instantaneous bulk moduli K^* as $K^* = K^*(e) = 1/2K_e(1+e)/e$. The 'one-half' was introduced to have $K^* = K_e$ when $e = 1$. Hence, the void ratio is taken into account by writing

$$p_{\mathrm{cr}} = p_{\mathrm{r}} \sqrt[3/2]{\frac{3K_e E_{\mathrm{c}}(1+e)}{8\vartheta p_{\mathrm{r}} e}}. \tag{5.9}$$

The last expression presents a factor that does not exist in Griffith's equation, the void ratio of the granular assembly, because voids are normally neglected in near-continuous solids. The result of this addition is our expectation that

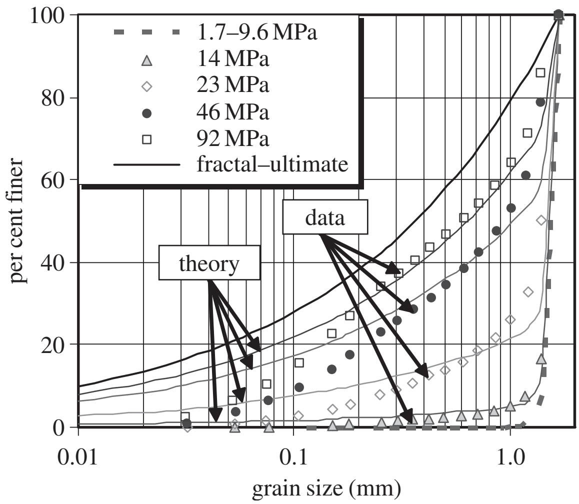

Figure 4. Theoretical versus experimental cumulative grain size distributions, for one-dimensional compression of silica sand (Nakata *et al.* 2001). The legend designates the corresponding effective vertical stresses.

agglomerates would break earlier only if they were initially loosely compacted. This phenomenon is well documented experimentally in the literature (e.g. Nakata *et al.* 2001).

The above result, however, is concerned only with the critical comminution pressure, before any crushing occurs. After fragmentation, the grain size distribution evolves and B grows ($\delta B>0$). This is what we refer to as 'fracture propagation'. The breakage condition in equation (5.6) requires that $\vartheta\Psi_{\mathrm{cr}}/E_{\mathrm{c}}=(1-\vartheta B)(1-B)^{-2}$, and since ϑ is a constant between zero and unity, then as crushing evolves and B gets closer to unity, more strain energy is required for further crushing. We write

$$p_{\mathrm{cr}} = p_{\mathrm{r}}\sqrt[3/2]{\frac{3K_e E_{\mathrm{c}}(1-\vartheta B)(1+e)}{8\vartheta p_{\mathrm{r}}(1-B)^2 e}}, \tag{5.10}$$

suggesting that the critical pressure increases with breakage. This explains, both rigorously and physically, what was known for many years in the soil mechanics community as isotropic hardening (Roscoe & Burland 1968; Schofield & Wroth 1968).

The applicability of this hardening is examined by analysing recent published experimental results from one-dimensional compression tests on silica sand (Nakata *et al.* 2001). The evolution of the grain size distribution as a function of the applied vertical stress is plotted in figure 4 together with the theoretical prediction using the fractional identity assumption. For each curve, we use equation (2.1) to produce the theoretical prediction, and measure B.

The different curves in figure 4 correspond to different normal effective vertical stresses σ_{v} (as designated in the figure legend). Assuming that the lateral/axial effective stress ratio is given by $k_0=0.35$, the corresponding pressures are estimated using $p=0.566\sigma_{\mathrm{v}}$. As mentioned before, the current theoretical analysis considers only the (active) breakage dissipation, ignoring the (passive) plastic dissipation from rearrangement of particles and friction. The plasticity effect could, in fact, be accounted for by adopting the coupling breakage–

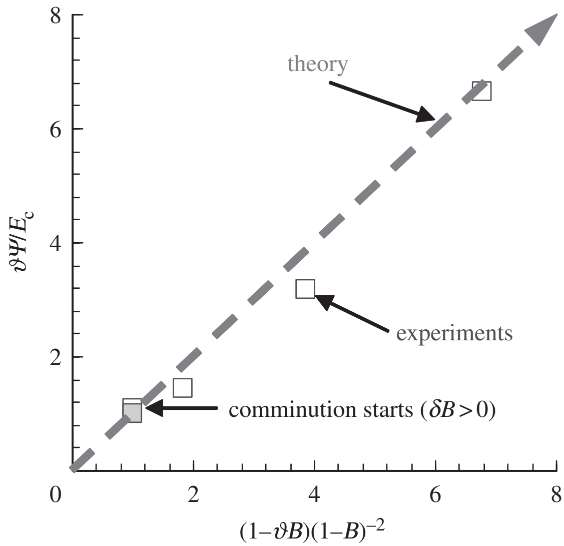

Figure 5. Theoretical versus experimental breakage hardening in one-dimensional compression of silica sand (Nakata *et al.* 2001), validating the fracture/breakage propagation criterion for brittle granular matter.

plasticity formulation of Einav (2007), and this reveals the famous linear relation between the logarithms of the void ratio and pressure by introducing plastic strains. For simplicity, however, the stored energy is measured via $\Psi \approx 4p_r(p/p_r)^{1.5}/3K^*(e)$. Therefore, the plasticity effect is indirectly accounted for experimentally by considering the continuous changes in the void ratio e during the measurement of Ψ. Using $\vartheta=0.87$, directly derived from the initial and ultimate distributions, the relation between breakage and isotropic hardening is examined by plotting the normalized energy $\vartheta\Psi/E_c$ versus $(1-\vartheta B)(1-B)^{-2}$ in figure 5. According to equation (5.6), there should be a one-to-one correlation, so the net effect of the parameters E_c and K_e and the initial void ratio is taken to satisfy this condition when breakage first occurs (i.e. $\vartheta\Psi_{cr}/E_c$ is taken unity when $B=0$). Theoretically, this one-to-one correlation should be maintained irrespective of B, and this is nicely shown in figure 5.

6. Critical comminution shear stress

Einav (2007*b*) suggested that when shear stresses are involved in the comminution process, the yielding condition of brittle granular materials may be expressed by

$$y_B = \left(\frac{E_B}{E_c}(1-B)^2\right)^2 + \left(\frac{q}{Mp}\right)^2 - 1 \leq 0. \tag{6.1}$$

Alternatively, similar arguments can lead to the following expression:

$$y_B = \frac{E_B}{E_c}(1-B)^2 + \left(\frac{q}{Mp}\right)^2 - 1 \leq 0, \tag{6.2}$$

which we adopt here simply because the mathematical arguments tend to be cleaner. We highlight that both expressions implicitly consider the active role that breakage takes in pure compression, since when $q=0$ the above equations reduce to equation (4.2), and the evolution of the breakage is taken as before. When some amount of shear is imposed, these new equations deviate from the ideal conditions, and the role of breakage evolution is gradually reducing and persistent frictional shear evolves with no grain crushing. Instead, grain crushing is developed from surface abrasion. As B tends to unity, and grain abrasion stops, both of the above expressions predict the Mohr–Coulomb friction law given by $q=Mp$.

According to equation (5.3), we can write

$$y_{\mathrm{B}} = \frac{\vartheta \Psi (1-B)^2}{E_{\mathrm{c}}(1-\vartheta B)} + \left(\frac{q}{Mp}\right)^2 - 1 \leq 0. \tag{6.3}$$

To derive the yielding condition in stress space, the stored energy Ψ has to be replaced by the stresses. For example, assuming linear elasticity for both shear and compression deformations, we have $\Psi = p^2/2K + 3q^2/2G$, where G is the shear modulus. This gives the following expression:

$$y_{\mathrm{B}} = \frac{\vartheta (Gp^2 + 3Kq^2)(1-B)^2}{2E_{\mathrm{c}}GK(1-\vartheta B)} + \left(\frac{q}{Mp}\right)^2 - 1 \leq 0. \tag{6.4}$$

The critical shear stress under the action of comminution is therefore expressed as follows:

$$q_{\mathrm{cr}} = \pm Mp \sqrt{\frac{2E_{\mathrm{c}}GK(1-\vartheta B) - \vartheta Gp^2(1-B)^2}{2E_{\mathrm{c}}GK(1-\vartheta B) + 3M^2p^2\vartheta K(1-B)^2}}. \tag{6.5}$$

We see that when breakage reaches its limit, we obtain the Mohr–Coulomb failure criterion. In these conditions, breakage ceases to play any role, and the energy is dissipated purely from grain sliding and rolling. The evolution of the yield surface, in equation (6.4), is given as a function of the breakage value B in figure 6.

The initial comminution pressure (according to equation (5.7)) is simply the intercept of the initial yield surface along the compression line ($p=0$), and the growth of this intercept value represents the isotropic hardening property of the material, as in critical state soil mechanics models (Roscoe & Burland 1968; Schofield & Wroth 1968). However, unlike critical state soil models, where the Mohr–Coulomb failure criterion is implicitly given as the product of the flow rule to the yield surface, here the Mohr–Coulomb failure criterion is simply the breakage yield surface itself when $B=1$. In critical state soil mechanics, the shape and size of the yield surface do not depend on the elastic properties of the granular material, but the current formulation predicts that they are, in accord with fracture mechanics.

7. Conclusions

Critical phenomena in nature often share similar properties. According to Griffith's energy analysis of the critical conditions for opening a crack immersed in a solid plate, the *critical tensile stress* depends on (i) the material via the solid's elastic Young's modulus E, (ii) its critical strain energy release rate G_{c}, and (iii) the geometry of the crack via its length $2a$. In this paper we adopt the energy

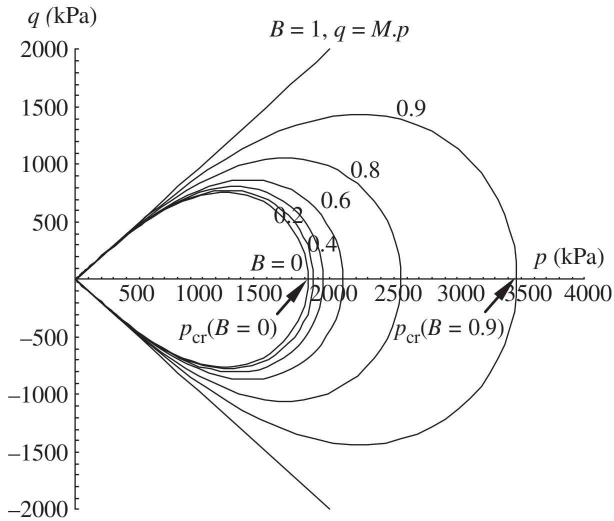

Figure 6. Evolution of the critical comminution yield surface due to breakage growth ($\vartheta = 0.9$, $K = 30\,000$ kPa, $G = 10\,000$ kPa, $E_c = 50$ kPa, $M = 1$).

formulation of breakage mechanics and find that the *critical comminution pressure* in confined brittle granular matter depends on similar properties. Young's modulus E is replaced by the bulk modulus K, Griffith's critical strain energy release rate constant G_c is replaced by the critical breakage energy constant E_c and, finally, the initial crack geometry $2a$ is replaced by $\vartheta = 1 - \langle d^2 \rangle_u / \langle d^2 \rangle_0$, where $4\pi \langle d^2 \rangle_0$ and $4\pi \langle d^2 \rangle_u$ denote the initial and ultimate statistical averaged surface areas of the grains. In addition, the critical comminution pressure depends further on the agglomerate porosity (or void ratio)—a property that is normally neglected in analysing near-continuous solids. As the granular material gets denser, and the coordination number increase, the bulk modulus of the granular matter increases. The above-mentioned critical comminution pressure was derived for pure isotropic loading conditions. To account for possible shear stresses, results have also been derived by combining the above model with the celebrated Mohr–Coulomb's failure criterion. The fundamental difference between the current formulation and critical state soil mechanics has been noted. This step in understanding enables for the first time to apply the principles of fracture mechanics to all disciplines dealing with confined particles comminution such as geophysics, geology, geotechnical engineering, mineral processing, agriculture and food industry, pharmaceuticals and powder technology.

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References

Aharonov, E. & Sparks, D. 1999 Rigidity phase transition in granular packings. *Phys. Rev. E* **60**, 6890–6896. (doi:10.1103/PhysRevE.60.6890)

Bak, P., Tang, C. & Wiesenfeld, K. 1987 Self-organized criticality: an explanation of $1/f$ noise. *Phys. Rev. Lett.* **59**, 381–384. (10.1103/PhysRevLett.59.381)

Corwin, E. I., Jaeger, H. M. & Nagel, S. R. 2005 Structural signature of jamming in granular media. *Nature* **435**, 1075–1078. (doi:10.1038/nature03698)

Einav, I. 2007*a* Breakage mechanics. Part I—theory. *J. Mech. Phys. Solids* **55**, 1274–1297. (doi:10.1016/j.jmps.2006.11.003)

Einav, I. 2007*b* Breakage mechanics. Part II—modelling granular materials. *J. Mech. Phys. Solids* **55**, 1298–1320. (doi:10.1016/j.jmps.2006.11.004)

Gordon, J. E. 1988 *The science of structures and materials*. New York, NY: Scientific American Library.

Griffith, A. A. 1921 The phenomena of rupture and flow in solids. *Phil. Trans. R. Soc. A* **221**, 163–198. (doi:10.1098/rsta.1921.0006)

Hardin, B. O. 1985 Crushing of soil particles. *J. Geotech. Eng. ASCE* **111**, 1177–1192.

Irwin, G. R. 1957 Analysis of stresses and strain near the end of a crack traversing a plate. *Trans. ASME Ser. E: J. Appl. Mech.* **24**, 361–364. [Discussion in *J. Appl. Mech.* 1958 **25**, 299–303.]

Kendall, K. 1978 The impossibility of comminuting small particles by compression. *Nature* **272**, 710–711. (doi:10.1038/272710a0)

Kick, F. 1883 Contribution to the knowledge of brittle materials. *Dinglers J.* **247**, 1–5.

Lynch, A. J. 1977 *Mineral crushing and grinding circuits. Their simulation, optimisation, design and control*. Amsterdam, The Netherlands: Elsevier Scientific Publishing Company.

McDowell, G. R., Bolton, M. D. & Robertson, D. 1996 The fractal crushing of granular materials. *J. Mech. Phys. Solids* **44**, 2079–2102. (doi:10.1016/S0022-5096(96)00058-0)

Nakata, Y., Hyodo, M., Hyde, A. F. L., Kato, Y. & Murata, H. 2001 Microscopic particle crushing of sand subjected to high pressure one-dimensional compression. *Soils Found.* **41**, 69–82.

Papamichos, E., Vardoulakis, I. & Ouadfel, H. 1993 Permeability reduction due to grain crushing around a perforation. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **30**, 1223–1229. (doi:10.1016/0148-9062(93)90099-Y)

Pestana, J. M. & Whittle, A. J. 1995 Compression model for cohesionless solids. *Géotechnique* **45**, 611–631.

Rice, J. R. 1978 Thermodynamics of the quasi-static growth of Griffith cracks. *J. Mech. Phys. Solids* **26**, 61–138. (doi:10.1016/0022-5096(78)90014-5)

Roscoe, K. H. & Burland, J. B. 1968 In *Engineering plasticity* (eds J. Heyman & F. A. Leckie), pp. 535–609. Cambridge, UK: Cambridge University Press.

Sammis, C. G., King, G. & Biegel, R. 1987 The kinematics of gouge deformations. *Pure Appl. Geophys.* **125**, 777–812. (doi:10.1007/BF00878033)

Schofield, A. N. & Wroth, C. P. 1968 *Critical state soil mechanics*. London, UK: McGraw-Hill.

Turcotte, D. L. 1986 Fractals and fragmentation. *J. Geophys. Res.* **91**, 1921–1926.

von Rittinger, R. P. 1867 *Textbook of mineral dressing*. Berlin, Germany: Ernest and Korn.

Walton, K. 1987 The effective elastic moduli of a random packing of spheres. *J. Mech. Phys. Solids* **35**, 213–226. (doi:10.1016/0022-5096(87)90036-6)

Wood, D. M. 2006 Geomaterials with changing grading: a route towards modelling. In *Int. Symp. on Geomechanics and Geotechnics of Particulate Media, IS-Yamaguchi, Ube, Japan, 12–14 September 2006*, pp. 313–316.