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On convexity, normality, pre-consolidation pressure, and singularities in modelling of granular materials

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Abstract The issues of convexity, normality, pre-consolidation pressure, and singularities of yield surfaces are discussed in the context of granular materials and soil mechanics. We approach those subjects from a rather unusual direction, by expressing yield surfaces in strain space. It is shown that the convexity assumption in strain space is justified when the elastic behaviour is linear, but not otherwise. As the effective bulk modulus of granular matter is generally pressure dependent, strain space yield surfaces are non-convex. However, strain space non-convexity does not necessarily violate the laws of thermodynamics, and by acknowledging that, arguments in favor of strain space elasto-plasticity could be made. We then define the pre-consolidation pressure directly using the total volumetric strain. The new definition offers to combine the advantages of the classical definition based on the void-ratio and a theoretically consistent definition using the plastic volumetric strain. It also allows removing singularities that may occur due to a zero denominator in the definition of the non-negative plasticity multiplier.

Keywords Plasticity · Thermo-mechanics · Convexity · Pre-consolidation pressure · Pressure dependent materials · Granular materials

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1 Introduction

Over the years, two general families of elasto-plasticity formulations have been adopted: the conventional stress space framework [1], and the more recent strain space framework [2]. The first approach is based on the existence of a yield surface in stress space, and the second on the existence of a yield surface in strain space. In both cases the convexity of the relevant yield surface is often postulated. This paper is initially concerned with the validity of the convexity assumption in strain space formulations, but subsequently a discussion is made with reference to the form of the relevant plasticity flow rule. We then highlight yet another space, the dissipative stress space, in which yield surfaces of elasto-plastic models are always convex and can be formulated via thermo-mechanical procedures. Throughout the text, special attention is given to the class of pressure dependent materials, such as various types of foams, biomaterials and granular assemblies. However, here we focus on geomaterials and a discussion of practical soil mechanics concepts is made using the definitions from the various formulations, with emphasis given to theoretical consistency and computational stability.

The use of the strain as a primary state variable in plasticity formulations has some history for soils, and such use has usually been accompanied by its own set of difficulties [3, 4]. Whereas stress space yield surfaces are commonly employed in geotechnical research and almost routinely used in design, to date strain space yield surfaces have found far less applications. The main reason has probably been to do with the fact that the geometrical interpretation of yield surfaces in stress space is much more intuitive. Initially, models were based on pure stress space plasticity concepts, but starting from the early sixties idealized models that were developed by the Cambridge School [5–7] used a critical-state locus in combined stress/void-ratio space, with the void-ratio determining the size of the yield surface when it is projected onto stress space. In other words the pre-consolidation pressure is taken as a function of the void ratio. As void-ratio is related to the volumetric strain, the actual size of the yield surface is linked to the amount of volumetric strain. The conclusion is that the actual yield surface in these models is fully expressed in a sort of mixed stress–strain space. Those critical state models have gained momentum ever since, but slowly researchers have started to question the theoretical validity of some of the underlying assumptions. One aspect, relevant to this paper, which was criticized using dimensional analysis and thermomechanical arguments, is that the void ratio can not be used explicitly, but can rather be implicitly included within the logarithm of the specific volume [e.g. 8–11]. As for the size of the yield surface, one alternative that is valid from the thermomechanical stand point is to take the pre-consolidation pressure as a function of the plastic volumetric strain rather than the void ratio [e.g. 12–15]. There is, however, a significant conceptual difference between these entities, because within the yield surface the plastic volumetric strain is constant while the void ratio changes. In other words, the stress space projection of the yield surface is inert when the pre-consolidation pressure is a function of the plastic volumetric strain. On the other hand, the yield surface effectively changes size in the void ratio model. Alternatively, as proposed here, the size of the yield surface could be determined using the *total* volumetric strain, preserving the inherent compressibility properties of the classical void ratio approach but also in a theoretically consistent manner, as we show here. Because volumetric strain is intrinsic in the definition of the yield surface, the validity of the convexity assumption of yield surfaces in strain space must be explored in the context of geomaterials.

In this paper it is shown that the convexity assumption in strain space is justified when the elastic behaviour is linear, but not otherwise. Although an earlier paper on strain space by Naghdi and Trapp [16] pointed out that strain space yield surfaces may be non-convex in the case of non-linear-elastic/plastic materials (but not following the same rigorous procedure we take here), this fact seems to have been ignored in some subsequent works [e.g. 17–19]. Strictly speaking, modern thermomechanics may even permit the situation of concave stress space yield surfaces [11], but apart from a limited amount of experimental evidence to support this exceptional possibility, stress space yield surfaces of geomaterials are mostly assumed to be convex [e.g. 20, 21]. It will be demonstrated that the strain space convexity postulation is generally violated by granular materials which exhibit non-linear elasticity due to the Hertzian contact law that governs the micro-interaction between grains [22].

2 Convexity postulates in stress and strain spaces

The flow rule for a given yield surface $g(\boldsymbol{\sigma}, \mathbf{e}_{\mathrm{p}}) = 0$ in stress space is said to be associated if:

$$\dot{\mathbf{e}}_{\mathrm{p}} = \Lambda \partial g / \partial \boldsymbol{\sigma} \tag{1}$$

where Λ is the non-negative plasticity multiplier and $\dot{\mathbf{e}}_{\mathrm{p}}$ is the rate of change of plastic strain, $\mathbf{e}_{\mathrm{p}}$. Based on the maximum work (or maximum dissipation) rate inequality [e.g. 23, 24], the following convexity requirement is normally postulated in stress space:

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\mathbf{e}}_{\mathrm{p}} \geq 0 \tag{2}$$

where $\boldsymbol{\sigma}^*$ denotes any permissible stress tensor within the yield surface; and $\boldsymbol{\sigma}$ is the current stress tensor.

If the yield surface is convex in stress space, then in associated plasticity the angle contained between the tensor $(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)$ and the plastic strain rate tensor $\dot{\mathbf{e}}_{\mathrm{p}}$ is acute in all cases (see Fig. 1a for a two-dimensional schematic representation). Thus the scalar product of the two tensors satisfies the inequality in Eq. (2). If the yield surface is not convex (as demonstrated in Fig. 1b) then we can always choose a new permissible stress $\boldsymbol{\sigma}^*$ so that the product of the two tensors is negative.

In the stress space elasto-plasticity framework it is common to assume that the total strain, $\mathbf{e}$, can be cumulatively decomposed as:

$$\mathbf{e} = \mathbf{e}_{\mathrm{e}} + \mathbf{e}_{\mathrm{p}} \tag{3}$$

where $\mathbf{e}_e$ is the elastic component of the strain, $\mathbf{e}$. In strain space elasto-plasticity an alternative cumulative decomposition is given by:

$$\boldsymbol{\sigma} = \mathbf{E} : \mathbf{e} - \boldsymbol{\sigma}_{\mathbf{r}} \tag{4}$$

This time it is the stress which is decomposed by eliminating the "relaxing stress" $\boldsymbol{\sigma}_{\mathbf{r}}$ from the "elastic stress"

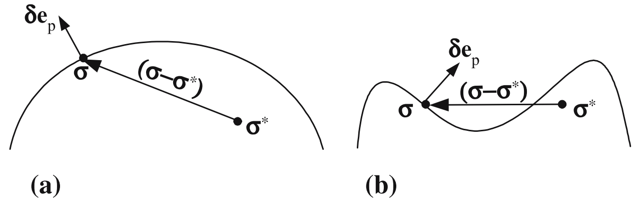

Fig. 1 **a** Convexity and **b** concavity of yield surfaces in strain space

(or "trial stress") **E**:**e**. The rate of change of the stress is then given by:

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\mathbf{e}} - \dot{\boldsymbol{\sigma}}_{\mathbf{r}} \tag{5}$$

Clearly, the cumulative decomposition over the stress is arguable to begin with as it requires the use of the linear stiffness tensor operator **E**. Ignoring this limitation Yoder [17] postulated, in a complementary manner to the stress-space convexity requirement, that provided a yield surface $f(\mathbf{e},\boldsymbol{\sigma}_{\mathbf{r}}) = 0$ exists in strain space, and provided the flow rule in this space is also associated and

$$\dot{\boldsymbol{\sigma}}_{\mathbf{r}} = \Lambda \partial f / \partial \mathbf{e} \tag{6}$$

it should be strictly convex, satisfying:

$$(\mathbf{e} - \mathbf{e}^*) : \dot{\boldsymbol{\sigma}}_{\mathbf{r}} \geq 0 \tag{7}$$

Figures 2a,b correspond to Figures 1a,b, showing the analogy between Yoder's [17] strain space hypothesis and the convexity requirement that follows from the maximum work hypothesis. It will be demonstrated, however, that this hypothesis may be easily violated when a non-linear law describes the elasticity relation within the yield surface, even when the maximum work hypothesis in the stress space holds.

Figure 3 illustrates the strain or stress decompositions (in one dimension) corresponding to the stress or strain space elasto-plasticity when the elasticity is linear.

3 The invalidaty of the convexity postulate in strain space

The Helmholtz free energy potential of an elasto-plastic material can be written generally as:

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_{\mathrm{e}}(\mathbf{e}_{\mathrm{e}}) + \boldsymbol{\Psi}_{\mathrm{p}}(\mathbf{e}_{\mathrm{p}}) \tag{8}$$

The stress is then given by:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}) = \boldsymbol{\Psi}'_{\mathrm{e}}(\mathbf{e}_{\mathrm{e}}) \tag{9}$$

If the elasticity rule is linear, i.e., the stress is a linear function of the strain, then according to Euler's theorem for first order homogeneous functions, $\boldsymbol{\sigma} = \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \mathbf{e}_{\mathrm{e}} = \boldsymbol{\Psi}''_{\mathrm{e}}(\mathbf{e}_{\mathrm{e}}) : \mathbf{e}_{\mathrm{e}}$. Since $\mathbf{e}_{\mathrm{e}}$ is of order 1 in the elastic

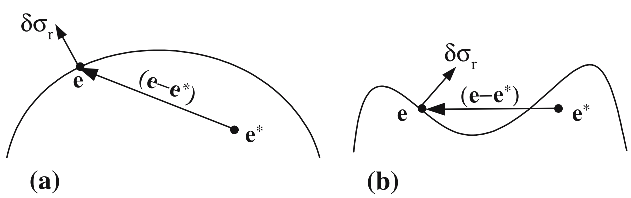

Fig. 2 **a** Convexity and **b** concavity of yield surfaces in strain space

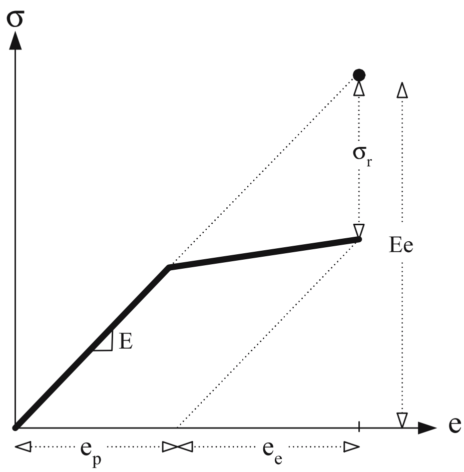

Fig. 3 Decompositions of strains ($e = e_{\mathrm{e}} + e_{\mathrm{p}}$) and stresses ($\boldsymbol{\sigma} = Ee - \boldsymbol{\sigma}_{\mathbf{r}}$) in linear-elastic/plastic models

strain, the tensor $\boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}})$ must be of order 0, i.e., it is a constant tensor known as the elasticity stiffness tensor **E**. Thus (using Eq. (3)), in this case only, the stress and its rate are given by:

$$\boldsymbol{\sigma} = \mathbf{E} : \mathbf{e}_{\mathrm{e}} = \mathbf{E} : \mathbf{e} - \mathbf{E} : \mathbf{e}_{\mathrm{p}} \tag{10}$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\mathbf{e}}_{\mathrm{e}} = \mathbf{E} : \dot{\mathbf{e}} - \mathbf{E} : \dot{\mathbf{e}}_{\mathrm{p}} \tag{11}$$

such that compared with Eqs. (4) and (5) the "relaxing stress" and its rate are given by:

$$\boldsymbol{\sigma}_{\mathbf{r}} = \mathbf{E} : \mathbf{e}_{\mathrm{p}} \tag{12}$$

$$\dot{\boldsymbol{\sigma}}_{\mathbf{r}} = \mathbf{E} : \dot{\mathbf{e}}_{\mathrm{p}} \tag{13}$$

Assuming that a convex yield surface exists in stress space $g(\boldsymbol{\sigma},\mathbf{e}_{\mathrm{p}})$ and satisfies Eq. (2), and since $\mathbf{e}_{\mathrm{p}} \equiv \mathbf{e}^*_{\mathrm{p}}$ by definition, then upon combination with Eqs. (10) and (13) we have:

$$\begin{aligned}\left[\mathbf{E} : (\mathbf{e} - \mathbf{e}^*)\right] : \dot{\mathbf{e}}_{\mathrm{p}} &= (\mathbf{e} - \mathbf{e}^*) : \left[\mathbf{E} : \dot{\mathbf{e}}_{\mathrm{p}}\right] \\ &= (\mathbf{e} - \mathbf{e}^*) : \dot{\boldsymbol{\sigma}}_{\mathbf{r}} \geq 0\end{aligned} \tag{14}$$

such that Yoder's [17] strain space convexity postulate in Eq. (7) is in fact validated. However, this was proven only for linear-elastic/plastic materials.

If the elasticity rule is non-linear, i.e., the stress is generally a non-linear function of the strain, $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}\prime(\mathbf{e}_{\mathrm{e}}) : \mathbf{e}_{\mathrm{e}}$, one can only adopt the more general relation (9) for the stress. Then the stress rate cannot be written as in Eq. (11), but only in the more general form:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \dot{\mathbf{e}}_{\mathrm{e}} = \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \dot{\mathbf{e}} - \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \dot{\mathbf{e}}_{\mathrm{p}} \tag{15}$$

It is still possible to define the rate of change of the "relaxing stress" by:

$$\tilde{\boldsymbol{\sigma}}_{\mathbf{r}} = \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \dot{\mathbf{e}}_{\mathrm{p}} \tag{16}$$

This time the tilde sign '∼' is introduced over $\boldsymbol{\sigma}_{\mathbf{r}}$, replacing the more common superimposed 'dot' sign in representing the rate, as in this case $\boldsymbol{\sigma}_{\mathbf{r}}$ is not a state variable (as the above expression is non-integrable). Equation (14) will then take only the more general form:

$$\left(\boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}) - \boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}^{*})\right) : \dot{\mathbf{e}}_{\mathrm{p}} \geq 0 \tag{17}$$

Conversely, combining the convexity requirement in Eq. (14) and the definition of the relaxing stress rate in Eq. (16) (and since $\mathbf{e}_{\mathrm{p}} \equiv \mathbf{e}_{\mathrm{p}}^{*}$):

$$\left(\boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \mathbf{e}_{\mathrm{e}} - \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \mathbf{e}_{\mathrm{e}}^{*}\right) : \dot{\mathbf{e}}_{\mathrm{p}} \geq 0 \tag{18}$$

Only when $\boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}})$ is a linear function of $\mathbf{e}_{\mathrm{e}}$, does the equality $\boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}) = \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) : \mathbf{e}_{\mathrm{e}}$ hold, in which case (17) and (18) are parallel requirements. Thus convexity may be violated in strain space in any other situation.

4 The strain space flow rule: thermodynamics perspective

Both of the stress and strain convexity requirements given in sect. 2 were originally limited to materials with an associated flow rule. One of the main advantages of thermo-mechanical models is that flow rules can be treated consistently and this has already been demonstrated for yield surfaces in stress space [25]. In this section the flow rule will be explored in the context of the counterpart strain space. To this end we adopt the additional thermo-mechanical notion of the 'rate of dissipation'.

The rate of dissipation can be expressed as a function of the strain, plastic strain and its rate, i.e., $\tilde{\boldsymbol{\Phi}} = \tilde{\boldsymbol{\Phi}}(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \dot{\mathbf{e}}_{\mathrm{p}}) \geq 0$.[1] According to Euler's theorem for homogeneous functions of order 1, in rate independent materials $\tilde{\boldsymbol{\Phi}} = \boldsymbol{\chi} : \dot{\mathbf{e}}_{\mathrm{p}}$ where we have used the so-called dissipative stress $\boldsymbol{\chi} = \partial\tilde{\boldsymbol{\Phi}}(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \dot{\mathbf{e}}_{\mathrm{p}})/\partial\dot{\mathbf{e}}_{\mathrm{p}}$. Based on thermodynamics for isothermal rate independent materials the sum of the rates of the Helmholtz free energy and of the dissipation should equate to the external work, such that via the use of Eq. (3):

$$\left(\boldsymbol{\sigma} - \boldsymbol{\Psi}_{\mathrm{e}}'(\mathbf{e}_{\mathrm{e}})\right) : \dot{\mathbf{e}}_{\mathrm{e}} + \left(\boldsymbol{\sigma} - \boldsymbol{\Psi}_{\mathrm{p}}'(\mathbf{e}_{\mathrm{p}}) - \boldsymbol{\chi}\right) : \dot{\mathbf{e}}_{\mathrm{p}} = 0 \tag{19}$$

Thus, according to Ziegler's orthogonality condition:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}) = \boldsymbol{\Psi}_{\mathrm{e}}'(\mathbf{e}_{\mathrm{e}}) \tag{20}$$

$$\boldsymbol{\chi} = \boldsymbol{\chi}(\mathbf{e}_{\mathrm{e}}, \mathbf{e}_{\mathrm{p}}) = \boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}) - \boldsymbol{\Psi}_{\mathrm{p}}'(\mathbf{e}_{\mathrm{p}}) \tag{21}$$

[1] The tilde sign over was $\tilde{\boldsymbol{\Phi}}$ introduced rather a more common superimposed 'dot' sign to represent rate, as there is no such thing as a state function "dissipation $\boldsymbol{\Phi}$".

A yield surface in *dissipative stress space* $\boldsymbol{\chi}$ could be defined from the properties of the degenerated special case of Legendre transformation. This transformation effectively replaces the plastic strain rates in the (first order homogeneous) rate of dissipation potential, by $\boldsymbol{\chi}$:

$$\Lambda y(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \boldsymbol{\chi}) \equiv \boldsymbol{\chi} : \dot{\mathbf{e}}_{\mathrm{p}} - \tilde{\boldsymbol{\Phi}}(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \dot{\mathbf{e}}_{\mathrm{p}}) = 0 \tag{22}$$

An important property of this transformation is that an associated flow rule may be defined in the dissipative stress space, by differentiating this quantity by $\boldsymbol{\chi}$:

$$\dot{\mathbf{e}}_{\mathrm{p}} = \Lambda \partial y/\partial \boldsymbol{\chi} \tag{23}$$

Collins and Houlsby [25] demonstrated the significance of defining the flow rule in *dissipative stress space,* by producing consistently associated or non-associated flow rules in *real stress space*. In a similar manner it is possible to use the same definition of a dissipative stress space yield surface in producing consistently associated or non-associated flow rules in *real strain space*.

Since during yielding $\tilde{\boldsymbol{\Phi}} = \boldsymbol{\chi} : \dot{\mathbf{e}}_{\mathrm{p}} \geq 0$, or $\boldsymbol{\chi} : \frac{\partial \mathbf{y}}{\partial \boldsymbol{\chi}} \geq 0$, the yield surface in the dissipative stress space is convex and contains the origin. The yield surface in strain space is defined by substituting $\boldsymbol{\chi} = \boldsymbol{\chi}(\mathbf{e}_{\mathrm{e}}, \mathbf{e}_{\mathrm{p}})$ and the strain decomposition, Eq. (3), into y:

$$y(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \boldsymbol{\chi}) \equiv f(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \boldsymbol{\sigma}(\mathbf{e}_{\mathrm{e}}) - \boldsymbol{\Psi}_{\mathrm{p}}'(\mathbf{e}_{\mathrm{p}})) = f(\mathbf{e}, \mathbf{e}_{\mathrm{p}}) \tag{24}$$

At this stage the "relaxing stress" concept is not employed, as in the general case of non-linear-elastic/plastic material only its increment can be defined. Differentiating the strain space yield surface $f(\mathbf{e}, \mathbf{e}_{\mathrm{p}})$ by the strain gives:

$$\frac{\partial f}{\partial \mathbf{e}} = \frac{\partial y}{\partial \boldsymbol{\chi}} \boldsymbol{\sigma}'(\mathbf{e}_{\mathrm{e}}) + \frac{\partial y}{\partial \mathbf{e}} \tag{25}$$

Multiplying by Λ and using Eqs. (16) and (23) gives the rate of change of the relaxing stress:

$$\tilde{\boldsymbol{\sigma}}_{\mathbf{r}} = \Lambda\left(\frac{\partial f(\mathbf{e}, \mathbf{e}_{\mathrm{p}})}{\partial \mathbf{e}} - \frac{\partial y(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \boldsymbol{\chi})}{\partial \mathbf{e}}\right) \tag{26}$$

This shows that the "relaxing stress increment" $\tilde{\boldsymbol{\sigma}}_{\mathbf{r}}$ is perpendicular to the strain space yield surface $f(\mathbf{e}, \mathbf{e}_{\mathrm{p}})$ (as required by Yoder's [17] normality assumption (6)), only if the dissipative yield surface is independent of the strain, $\mathbf{e}$, i.e., $y = y(\mathbf{e}_{\mathrm{p}}, \boldsymbol{\chi})$. In terms of computations, only $\mathbf{e}_{\mathrm{p}}$ is required as "$\boldsymbol{\sigma}_{\mathbf{r}}$" is not a state variable. The plastic strain is found by integrating its rate of change $\dot{e}_p$ (given by Eq. (23)). If $y = y(e, e_p, \chi)$ is also a function of the strain $\mathbf{e}$, then due to the extra term $\partial y(\mathbf{e}, \mathbf{e}_{\mathrm{p}}, \boldsymbol{\chi})/\partial \mathbf{e}$ in Eq. (26), Eq. (7) might be violated even in the case of linear elasticity.

The following study is aimed to explore the computational consequences when using two specialized forms of yield surfaces: one independent of total strain, $y = y(\mathbf{e}_\mathrm{p}, \boldsymbol{\chi})$, or one independent of plastic strain $y = y(\mathbf{e}, \boldsymbol{\chi})$.

4.1 Computational aspects linked with $y = y(\mathbf{e}_\mathrm{p}, \boldsymbol{\chi})$

In this case the consistency condition of the yield surface is given by:

$$\dot{y} = \frac{\partial y}{\partial \mathbf{e}_\mathrm{p}} : \dot{\mathbf{e}}_\mathrm{p} + \frac{\partial y}{\partial \boldsymbol{\chi}} : \dot{\boldsymbol{\chi}} = 0 \tag{27}$$

The non-negative multiplier can be derived by substituting Eqs. (20), (21) and (23):

$$\Lambda = \frac{\dfrac{\partial y}{\partial \boldsymbol{\chi}} : \boldsymbol{\Psi}_{\mathbf{e}''}(\mathbf{e}_\mathrm{e}) : \dot{\mathbf{e}}}{\dfrac{\partial y}{\partial \boldsymbol{\chi}} : \left(\boldsymbol{\Psi}_{\mathrm{e}''}(\mathbf{e}_\mathrm{e}) + \boldsymbol{\Psi}_{\mathrm{p}''}(\mathbf{e}_\mathrm{p})\right) : \dfrac{\partial y}{\partial \boldsymbol{\chi}} - \dfrac{\partial y}{\partial \mathbf{e}_\mathrm{p}} : \dfrac{\partial y}{\partial \boldsymbol{\chi}}} \tag{28}$$

Let us coin the term, the *singularity function,* expressing the denominator as:

$$\Gamma = \frac{\partial y}{\partial \boldsymbol{\chi}} : \mathbf{A} : \frac{\partial y}{\partial \boldsymbol{\chi}} - \frac{\partial y}{\partial \mathbf{e}_\mathrm{p}} : \frac{\partial y}{\partial \boldsymbol{\chi}} \tag{29}$$

where $\mathbf{A}$ is given by:

$$\mathbf{A} = \boldsymbol{\Psi}''_\mathrm{e}(\mathbf{e}_\mathrm{e}) + \boldsymbol{\Psi}''_\mathrm{p}(\mathbf{e}_\mathrm{p}) \tag{30}$$

The *singularity function* becomes the *singularity surface* when $\Gamma = 0$, giving all possible combinations where the denominator in Λ becomes zero, hence leading to infinite Λ. Computationally, this situation is of course unwanted.

4.2 Computational aspects linked with $y = y(\mathbf{e}, \boldsymbol{\chi})$

In this case the consistency condition is given by:

$$\dot{y} = \frac{\partial y}{\partial \mathbf{e}} : \dot{\mathbf{e}} + \frac{\partial y}{\partial \boldsymbol{\chi}} : \dot{\boldsymbol{\chi}} = 0 \tag{31}$$

giving:

$$\Lambda = \frac{\left(\dfrac{\partial y}{\partial \boldsymbol{\chi}} : \boldsymbol{\Psi}_{\mathrm{e}''}(\mathbf{e}_\mathrm{e}) + \dfrac{\partial y}{\partial \mathbf{e}}\right) : \dot{\mathbf{e}}}{\dfrac{\partial y}{\partial \boldsymbol{\chi}} : \left(\boldsymbol{\Psi}_{\mathrm{e}''}(\mathbf{e}_\mathrm{e}) + \boldsymbol{\Psi}_{\mathrm{p}''}(\mathbf{e}_\mathrm{p})\right) : \dfrac{\partial y}{\partial \boldsymbol{\chi}}} \tag{32}$$

The *singularity function* is thus defined by:

$$\Gamma = \frac{\partial y}{\partial \boldsymbol{\chi}} : \mathbf{A} : \frac{\partial y}{\partial \boldsymbol{\chi}} \tag{33}$$

where compared with Eq. (29), the last term is omitted. From the definition of positive definite matrices, for positive definite **A,** Γ must be positive. Therefore, from a computational standpoint the above formulation is advantageous compared to the previous alternative.

5 Various definitions of pre-consolidation pressure

In critical state soil mechanics, the projection of the yield surface on the $q-p$ stress space intersects the isotropic line ($q = 0$) at a value of the mean effective stress, p, called the "pre-consolidation pressure", p_c. This quantity describes the asymptotic behaviour of the model under isotropic ($q = 0$) normal compression conditions ($p = p_\mathrm{c}$) using some kinematic variable. Another asymptotic feature of the critical state soil models allows predicting a failure line under shearing conditions, often in terms of the linear Mohr-Coulomb criterion, $q = Mp$, with M being the friction parameter (see Fig. 4a).

In the early developments of critical state soil models [6, 7], the pre-consolidation pressure of remoulded clays was expressed purely using the specific volume $v = 1 + e$ (with e being the void ratio). In these models the typical isotropic compression curve ($q = 0$), under loading, unloading and reloading can be expressed in $\log(p) - v$ space by Fig. 4b, with κ designating the elasticity compressibility modulus, and λ the normal consolidation compression modulus. The original expression for the pre-consolidation pressure in those models is given by:

$$p_c = p_c(v) = p_\mathbf{r} \exp\left(\frac{v_\lambda - v}{\lambda}\right) \tag{34}$$

where $p_\mathbf{r}$ is a reference quantity with the dimensions of stress, necessary to establish dimensional consistency, and normally having the value 1 kPa. In this model the void ratio is related to the logarithm of the pressure. A more recent suggestion, which is supported from both dimensional and thermo-mechanical considerations [8–11] is to express linearity between the pre-consolidation pressure and the void ratio in log–log space, i.e., a linear relation between $\log(p_\mathrm{c})$ and $\log(v)$. From the definition of small strains, the increment of the volumetric strain is given by $de_\mathrm{v} = dv/v$, such that upon integration it is possible to define the volumetric strain with respect to the initial specific volume:

$$e_\mathrm{v} = \log(v_0/v) \tag{35}$$

in which case the volumetric strain is assumed zero when $v = v_0$. The normal compression line is linear with the slope $\lambda^* = \lambda/v$ in $\log(p_\mathrm{c}) - e_\mathrm{v}$ space. From Fig. 4c, prior to yielding, under isotropic conditions ($q = 0$) the elastic volumetric strain may be expressed as $e_v^\mathrm{e} = \kappa^* \log(p_c/p_0)(\kappa^* = \kappa/v)$, where p_0 is the initial mean effective stress. Under the same conditions the total volumetric strain is given by $e_v = \kappa^* \log(p_\mathrm{c}/p_0) + \lambda^* \log(p_\mathrm{c}/p_{c0})$. The plastic volumetric strain may be defined by subtracting the elastic volumetric strain from

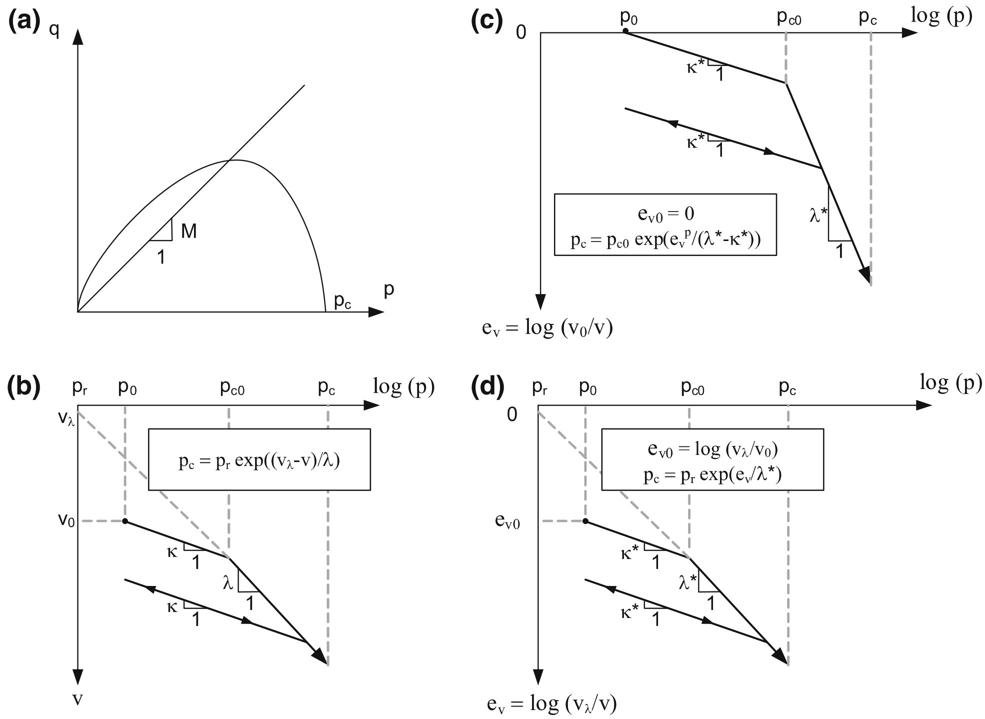

Fig. 4 **a** Typical yield surface projection in stress-space, **b–d** three alternative definitions of the pre-consolidation pressure p_c

the total strain, i.e., $e_v^p = (\lambda^* - \kappa^*)\log(p_c/p_{c0})$, thus the pre-consolidation pressure is given by [e.g., 12]:

$$p_c = p_c(e_v^p) = p_{c0}\exp\left(\frac{e_v^p}{\lambda^* - \kappa^*}\right) \tag{36}$$

There is, however, a significant conceptual difference between the last two expressions, as within the yield surface the plastic volumetric strain is constant while the void ratio changes. Furthermore, in the first model use is made of the reference kinematic configuration (i.e., the specific volume at the reference pressure, v_λ), while in the second model this parameter is replaced by the initial pre-consolidation pressure, p_{c0}. This variation is quite significant since the specific volume reference value is unique for a given soil, while the initial pre-consolidation pressure p_{c0} is dependent upon the history of loading.

The question is then, how can we make use of the advantages of both models? For that purpose our proposition is to express the size of the yield surface using the *total* volumetric strain (see Fig. 4d).

This time, the definition of the volumetric strain is taken from the reference configuration, thus the integration of the increment of the volumetric strain $de_v = dv/v$ gives:

$$e_v = \log(v_\lambda/v) \tag{37}$$

The initial volumetric strain is then $e_{v0} = log(v_\lambda/v_0)$. In this configuration we define the pre-consolidation pressure using the total volumetric strain:

$$p_c = p_c(e_v) = p_\mathbf{r}\exp\left(\frac{e_v}{\lambda^*}\right) \tag{38}$$

This expression is, in fact, a combination of the previous two: while the reference pressure is used as in the first model, the modified normal compression parameter λ^* is used as in the second model. In the next section we will explore the last two propositions from a thermo-mechanical perspective, but we will also endeavour to explore the convexity issues that arise when expressing yield surfaces in strain space.

6 Deriving critical state models from thermo-mechanics and expressing yield surfaces in strain space

6.1 Typical derivation of critical state model from thermodynamics

In critical state models, the Helmholtz free energy potential can be expressed as follows [26]:

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_e(\mathbf{e}_e) = \kappa^* p_\mathbf{r}\exp\left(e_v^e/\kappa^*\right) + \tfrac{3}{2}G e_s^{e^2} \tag{39}$$

The Helmholtz potential may be used to derive the evolution equations for the mean effective stress and shear stress by:

$$p = \partial\Psi_e/\partial e_v^e = p_{\mathbf{r}} \exp\left(e_v^e/\kappa^*\right) = \chi_p \tag{40}$$

$$q = \partial\Psi_e/\partial e_s^e = 3Ge_s^e = \chi_q \tag{41}$$

where the last equality in both equations is attributed to Eq. (21). In this model the elastic volumetric strain is related to the logarithm of the stress, but the shear strain is linearly related to the shear stress. Differentiating p with respect to the elastic volumetric strain suggests that in this model the bulk modulus is proportional to the mean effective stress and equal to p/κ^*, which will present consequences in terms of the non-convexity of the yield surface in strain space.

In order to complete the thermo-mechanical model, one has to establish a rate of dissipation potential, which in return allows deriving the yield surface, initially in dissipative stress space, but subsequently in stress or strain spaces. In this model, the dissipation is written as [25]:

$$\tilde{\Phi} = \frac{p_c(\bullet)}{2}\left(\sqrt{\dot{e}_v^{p2} + M^2\dot{e}_s^{p2}} + \dot{e}_v^p\right) \tag{42}$$

being dependent upon the definition of $p_c(\bullet)$, where '$\bullet$' stands either for the plastic volumetric strain, e_v^p, or the total volumetric strain, e_v. Using the degenerate special case of the Legendre transformation for first order homogeneous functions, the yield surface is initially given by:

$$y(\chi_p,\chi_q,\bullet) = \left(\chi_p - \tfrac{1}{2}p_y(\bullet)\right)^2 + \left(\chi_q/M\right)^2 - \left(\tfrac{1}{2}p_c(\bullet)\right)^2 = 0 \tag{43}$$

The rate of change of the plastic volumetric and shear strains is given by using the flow rule of Eq. (23), thus $\dot{e}_v^p = \Lambda\partial y/\partial\chi_p$ and $\dot{e}_s^p = \Lambda\partial y/\partial\chi_s$. Dividing the two eliminates the non-negative plasticity multiplier, Λ:

$$\frac{\dot{e}_v^p}{\dot{e}_s^p} = M^2\frac{\chi_p - \frac{1}{2}p_c(\bullet)}{\chi_q} \tag{44}$$

For brevity, let us denote the $\bullet = e_v^p$ model, as "Model ℵ", and the $\bullet = e_v^p$ model, as "Model ℶ". The yield surface in dissipative stress space (Eq. (43)) is represented in Fig. 5a for typical parameters ($G = 10\,\text{MPa}$, $\kappa^* = 0.02, \lambda^* = 0.2, M = 1.4$). To get compatibility between the two models, we use $p_{c0} = 100\,\text{kPa}$ in the first and $e_{v0} = \lambda^* \log(p_{c0}/p_{\mathbf{r}}) = 0.921$ in the second, with $p_{\mathbf{r}} = 1\,\text{kPa}$.

6.2 The plastic volumetric strain pre-consolidation model (Model ℵ)

This is the case in which the pre-consolidation pressure is given by Eq. (36), or in other words when $\bullet = e_v^p$ and $y(\chi_p,\chi_q,\bullet) = y(\chi_p,\chi_q,e_v^p)$.

The *singularity surface* is given by using Eq. (29):

$$\Gamma = (2p - p_c)^2\frac{p}{\kappa^*} + (2p - p_c)\frac{p_c p}{\lambda^* - \kappa^*} + \frac{12q^2}{M^4}G = 0 \tag{45}$$

In order to obtain the yield surface in stress space, the dissipative stresses are replaced by the true stress. The yield surface projection on stress-space is thus given by a conventional modified Cam clay (MCC) ellipse:

$$g(p,q,e_v^p) = \left(p - \tfrac{1}{2}p_c(e_v^p)\right)^2 + (q/M)^2 - \left(\tfrac{1}{2}p_c(e_v^p)\right)^2 = 0 \tag{46}$$

For the same set of parameters as we used in Fig. 5a, the above two surfaces are represented in Fig 5b. In this case, the singularity surface is well inside of the yield surface, excluding the singular conditions with $g = \Gamma = 0$.

The stress space yield surface can be converted into a strain space yield surface by using Eqs. (40) and (41), and the strain decomposition in Eq. (3):

$$f(e_v,e_s,e_v^p,e_s^p) = \left(p_{\mathbf{r}}\exp\left(\frac{e_v - e_v^p}{\kappa^*}\right) - \tfrac{1}{2}p_{c0}\exp\left(\frac{e_v^p}{\lambda^* - \kappa^*}\right)\right)^2 + \left(\frac{3G(e_s - e_s^p)}{M}\right)^2 - \left(\tfrac{1}{2}p_{c0}\exp\left(\frac{e_v^p}{\lambda^* - \kappa^*}\right)\right)^2 = 0 \tag{47}$$

This yield surface in strain space is represented in Fig. 5d, after normal compression of a virgin sample to $p_c = 100\,\text{kPa}$. In this case, the yield surface is all to the left, because no further volumetric strain is needed for yielding. It is shown that the yield surface has concave sections due to the non-linear elasticity rule.

6.3 The total volumetric strain pre-consolidation model (Model ℶ)

This is the case in which the pre-consolidation pressure is defined by Eq. (38), or in other words when $\bullet = e_v$.

The *singularity function* is given by using Eq. (33):

$$\Gamma = (2p - p_c)^2\frac{p}{\kappa^*} + \frac{12q^2}{M^4}G > 0 \tag{48}$$

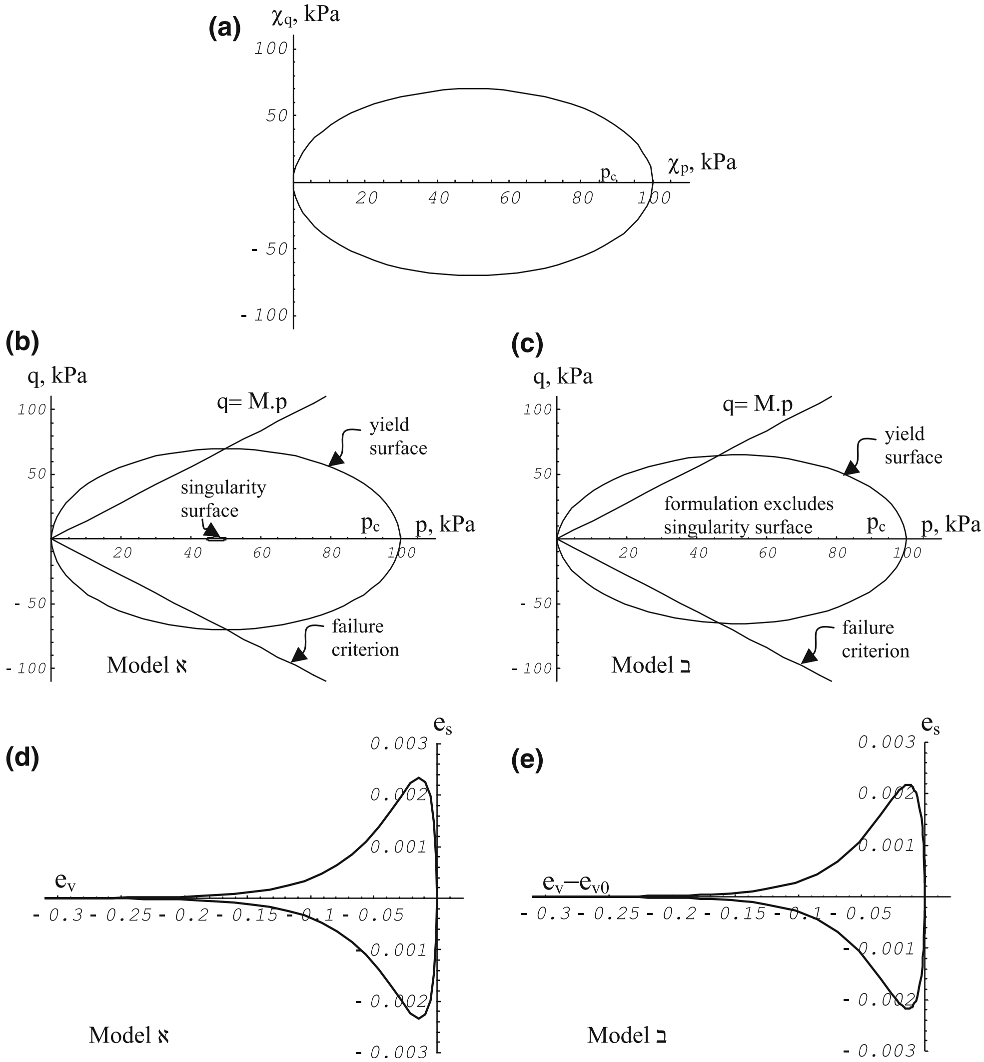

Fig. 5 a Yield surface in dissipative stress space for both Models ℵ and ℶ. **b** and **d** gives the stress, and strain space projections of the yield surface in Model ℵ, while (**c**), and (**e**) gives the stress, and strain space projections in Model ℶ (parameters: $G = 10\,\text{MPa}$, $\kappa^* = 0.02, \lambda^* = 0.2, M = 1.4$; $p_{c0} = 100\,\text{kPa}$; $e_{v0} = 0.921$; $p_{\mathbf{r}} = 1\,\text{kPa}$)

and since in Cam clay models we always have $p > 0$, we get $\Gamma > 0$.

Unlike the previous model, replacing the dissipative stresses by the true stress, leads to an unusual form of mixed stress–strain space yield surface $y_f^g = 0$:

$$y_f^g(p, q, e_v, e_s) = \left(p - \tfrac{1}{2}p_{\mathbf{r}} \exp\left(\frac{e_v}{\lambda^*}\right)\right)^2 + \left(\frac{q}{M}\right)^2 - p_r^2\left(\tfrac{1}{2}\exp\left(\frac{e_v}{\lambda^*}\right)\right)^2 = 0 \quad (49)$$

For a given constant e_{v}, the yield surface is still represented by a MCC ellipse. However, as volumetric strains can occur prior to yielding due to the elastic deformations, the pure stress-space projection of the yield surface is modified and given by eliminating e_{v} from the last expression, using Eqs. (40) and (3):

$$g(p, q, e_v^p) = \left(p - \tfrac{1}{2}p_{\mathbf{r}} \exp\left(\frac{\kappa^* \log(p/p_{\mathbf{r}}) + e_v^p}{\lambda^*}\right)\right)^2 + (q/M)^2 - \left(\tfrac{1}{2}p_{\mathbf{r}} \exp\left(\frac{\kappa^* \log(p/p_{\mathbf{r}}) + e_v^p}{\lambda^*}\right)\right)^2 = 0 \quad (50)$$

For the same set of parameters as we used in Fig. 5a, the above yield surface is represented in Fig. 5c. In this case, the singularity surface does not exist, as Γ is always positive.

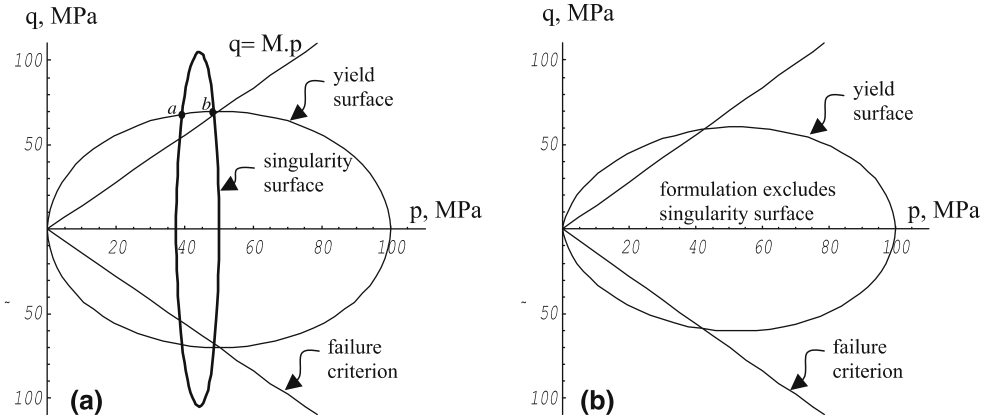

Fig. 6 Various surfaces in stress space of **a** Model ℵ and **b** Model ℶ (parameters: $G = 10\,\text{MPa}$, $\kappa^* = 0.02, \lambda^* = 0.2, M = 1.4$; $p_{c0} = 100\,\text{MPa}$)

It is shown that the yield surface in "Model ℶ" slightly deviates from the conventional MCC ellipse. Furthermore, the apex of this yield surface is not aligned with the failure surface. Although deviating from the conventional ellipse, the model still preserves the main aspects of critical state soil mechanics by expressing the asymptotic behaviour of the clay in isotropic compression and its failure criterion $q = M.p$ under shearing conditions.

To derive the yield surface projection on strain space, the true stresses in Eq. (49) are replaced by the strains using Eqs. (40) and (41):

$$f(e_v, e_s, e_v^p, e_s^p) = p_r^2\left(\exp\left(\frac{e_v - e_v^p}{\kappa^*}\right) - \frac{1}{2}\exp\left(\frac{e_v}{\lambda^*}\right)\right)^2 + \left(\frac{3G(e_s - e_s^p)}{M}\right)^2 - p_r^2\left(\frac{1}{2}\exp\left(\frac{e_v}{\lambda^*}\right)\right)^2 = 0 \qquad (51)$$

This yield surface in strain space is represented in Fig. 5e, for the same conditions as we used in Fig. 5d. The x-axis is represented by omitting the initial volumetric strain from the volumetric strain. As with the other model, in this space the yield surface has concave sections due to the non-linear elasticity rule.

6.4 Singularity surface

The singularity surface was never an issue in the previous analysis, as it was well inside the yield surface for the parameters given in Model ℵ . However, under extreme conditions, this may well change. For example, consider the case when $G = 10\,\text{MPa}$, $\kappa^* = 0.02, \lambda^* = 0.1, M = 1.4$, and the sample is loaded to a much higher pre-consolidation pressure $p_c = 100\,\text{MPa}$. For these conditions, the surfaces in Model ℵ and Model ℶ, are given in Fig. 6a,b.

As before, the formulation of Model ℶ excludes the existence of the singularity surface. However, in Model ℵ the singularity surface intersects the elliptical yield surface at points *a* and *b,* denoting $g = \Gamma = 0$, in which case the non-negative multiplier Λ becomes infinite by definition. In fact, this quantity is ill-defined whenever yielding occurs along the section $a - b$, as in those cases where $\Gamma < 0$.

7 Conclusions

This paper has presented non-standard techniques that enable interpretation of some existing soil mechanics models, but also the development of original mathematical forms of simple models. We demonstrate how yield surfaces of elasto-plastic models, which were originally presented in stress-space, could be analysed in strain space. It is shown that the convexity assumption in strain space is justified when the elastic behaviour is linear, but not otherwise. However, linear elasticity is not applicable for many materials, such as granular materials. As the effective bulk modulus of geomaterials is generally pressure dependent, strain space yield surfaces are non-convex. Another conclusion is that the "relaxing stress" tensor can not be regarded as a state variable, and only its rate of change can be identified if the elastic law of the model is non linear. The rate of change of the relaxing stress is perpendicular to the strain space yield surface only if the dissipative yield surface is independent of the total volumetric strain.

A proposition is made to explore a new form of expression for the pre-consolidation pressure, dependent upon the total volumetric strain rather than the plastic volumetric strain or the void-ratio. This means that in this model the rate of change of the relaxing stress is non-associated to the strain space yield surface. Similar to the plastic volumetric strain model, the total value of the relaxing stress itself can not be defined, as in both models the bulk modulus is pressure dependent. However, both are shown to be consistent from a thermo-mechanical stand point. Unlike the plastic strain model, but similar to the void ratio model, the total strain model accounts for the initial compressibility state of the soil such that the model's parameters are unique to a given soil regardless of the loading history. In addition, the model offers an advantageous computational treatment, where singularities are minimized. It should be mentioned, however, that taking the hardening as a function of the total volumetric strain requires further study, in particular with respect to the uniqueness of the critical state for certain loading histories and the softening behaviour in shear deformations.

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