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ON THE VALIDITY OF ELASTIC/PLASTIC DECOMPOSITIONS IN SOIL MECHANICS

Ian F. COLLINS*, Itai EINAV**

* Department of Engineering Science, University of Auckland, Private Bag 92019, Auckland, NEW ZEALAND, i.collins@auckland.ac.nz.
** Centre for Offshore Foundation Systems, University of Western Australia, 35, Stirling Highway, Crawley, Perth, WA 6009, AUSTRALIA, einev@cyllene.uwa.edu.au.

Abstract

This paper is concerned with the validity of decomposing the total strain into elastic and plastic components, and treating these components as “state variables”, which are independent of the previous stress history. It is shown that this procedure is justified when the micro-mechanical elastic behaviour is linear, but not otherwise. The corresponding decomposition of the strain or deformation rate is always valid. This result is important for soils since the micro-elastic behaviour is rarely linear. The arguments are presented in terms of spring-slider systems and continuum homogenization theory. Analyses of the corresponding decomposition of the free energy in thermomechanical analyses is also discussed.

Keywords: Elastoplasticity, geomechanics, thermomechanics, soil mechanics.

1.Introduction

In continuum elastoplasticity theory it is common to regard the strain as the sum of an elastic and plastic strain, and the constitutive functions, such as the instantaneous elastic moduli, free energy or yield function, are assumed to be functions of these two strain components. Similar decompositions are presumed for the strain (or deformation) rates. The validity of these procedures is seldom questioned. Furthermore in thermomechanical approaches it is common to regard the free energy as the sum of two functions, one dependent on the elastic strain and the other on the plastic strain. This hypothesis is often referred to as the “principle of separation of energies” (Ulm and Coussy, 2003) and such models are termed decoupled (Collins and Houlsby, 1997). In this paper we are concerned with the validity of these decompositions in the context of soil mechanics.

In formulating constitutive equations, the independent variables must be “state variables”, which describe the current state of a “continuum or representative volume element (RVE)”. Some of these variables may depend on the strain history of the element, such as the effective deviatoric strain, widely used in metal plasticity. In reality the stress and deformation fields are inhomogeneous within a typical RVE. This is particularly true for a soil or granular medium, where the development of force chains, results in a very wide spectrum of stress levels within an RVE. The continuum, elastic and plastic strains are some form of "average" of the recoverable and irrecoverable micro-strains occurring within the RVE. These averaging processes have been widely studied in the context of metal plasticity (Mroz, 1973, Maugin, 1992). A common feature of these theories is that the micro-elastic behaviour is assumed to be *linear*, an assumption, which is realistic for metals, but not for granular materials, where the micro-elastic strains result from the Hertzian contact of individual grains. Such behaviour results in non-linear stress-strain behaviour (Johnson,1965) This is readily observed at the continuum level, where the non-linear nature of the elastic deformations is well established (Oda and Iwashita, 1999).

In this paper the consequences of having non-linear elastic deformations at the micro-level for the formulation of constitutive laws at the continuum level is explored. This is achieved initially by considering systems of springs and sliders, which represent the micro-level deformations. This analysis is then generalized, using the standard techniques of homogenization theory.

2. Multi-mode systems with linear elasticity

In this section we consider a system of spring-sliders in parallel as shown in Fig.1. This system was introduced by (Masing, 1926; Iwan 1966, 1967), and recently further discussed by (Einav, 2004a). In such a system the strain $\bar{e}$ is the same in all legs in accord with the Voigt homogenization procedure (Maugin, 1992).

The free (strain) energy of the system is:

$$\overline{\Psi}[\bar{e}, e_i^{mp}] \equiv \tfrac{1}{2}\Sigma K_i {e_i^{me}}^2 = \tfrac{1}{2}\Sigma K_i (\bar{e} - e_i^{mp})^2 \quad (1)$$

where e_i^{me}, e_i^{mp} and K_i are the elastic and plastic strains, and stiffness in the i^{th} spring-slider element respectively. The total strain $\bar{e}$ and the internal variables e_i^{mp} uniquely

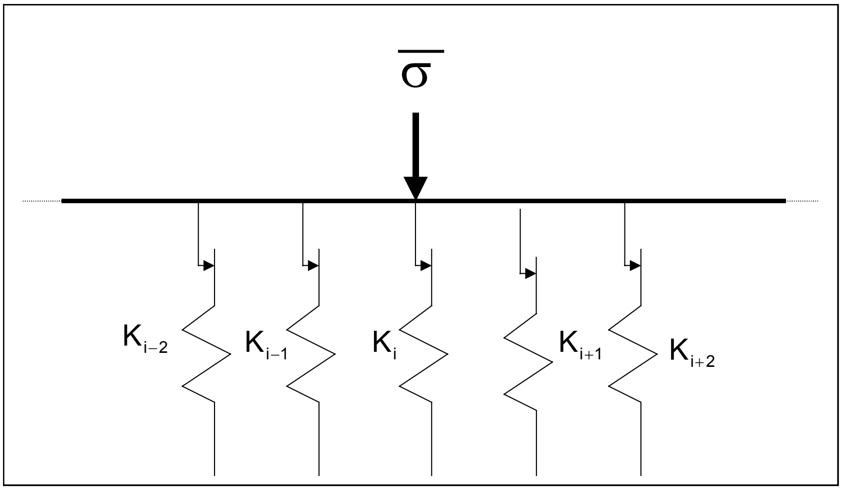

Fig 1. The multi-mode system of springs in parallel. For a nonlinear system the behaviour of each spring is defined by the free (strain) energy function. The total strain is the same across each "leg" of the system.

define the state of the system. We note that the free energy depends only on the elastic strains in the springs. As will be seen, this is not true at the system level, where the free energy depends on the plastic as well as the elastic strains. Similarly the total stress is given by:

$$\overline{\sigma}[\overline{e}, e_i^{mp}] \equiv \Sigma\sigma_i = \Sigma K_i e_i^{me} = \Sigma K_i(\overline{e} - e_i^{mp}) \tag{2}$$

We are adopting the convention that all system variables will have a superposed bar. The system elastic and plastic strain are defined to be respectively: (a) the strain pertaining when the plastic micro-strains in the springs are all zero, and (b) the strain at which the system stress is zero. Hence:

$$\overline{e}^E = \Sigma(K_i e_i^{me})/\Sigma K_i) = \overline{\sigma}/\overline{K}, \text{ and } \overline{e}^P = \Sigma(K_i e_i^{mp})/\overline{K} \tag{3}$$

where $\overline{K} \equiv \Sigma K_i$ is "total" system stiffness. It should be noted that, from (2), the sum of these two strains is the total strain as expected, and that these elastic and plastic strains are state variables, since they are determined by the current micro-strains. These system strains are not, in general, sufficient to *uniquely* determine the system. The exception is the "uni-modal" system consisting of one purely elastic leg, in parallel with a single elastic-plastic leg, which correspond to the classical soil models based upon the use of a single elastic and plastic strain variables, as in the critical state theories (Collins 2005b).

It is important to note that the plastic strain is a weighted average of the plastic strains in the sliders, and is not simply Σe_i^{mp}. This is because, although the system stress and elastic strain are zero in the unloaded state, the stresses and elastic strains in the individual springs are still nonzero in general. It follows from the previous equations that these residual stresses σ_i^r, and residual elastic strains e_i^{mr} have the properties:

$$\Sigma\sigma_i^r = \Sigma K_i e_i^{mr} = 0, \overline{e}^P = e_i^{mp} + e_i^{mr}, \text{ and so } e_i^{me} = \overline{e} - e_i^{mp} = \overline{e}^E + e_i^{mr} \tag{4}$$

These residual elastic strains can be viewed as the "deviation" plastic strains, since when they are all zero the plastic strains in each slider are all the same and equal to the system plastic strain $\overline{e}^P$. Although these residual strains are physically elastic, their magnitude is determined by the plastic strains as is clear from the second equation (4).

From (1) and (4) it follows that the total free energy of the system is:

$$\overline{\Psi}[\overline{e}, e_i^{mp}] = \tfrac{1}{2}\Sigma K_i e_i^{me^2} = \tfrac{1}{2}\Sigma K_i(\overline{e}^E + e_i^{mr})^2 = \tfrac{1}{2}\overline{K}\overline{e}^{E^2} + \tfrac{1}{2}\Sigma K_i e_i^{mr^2} \tag{5}$$

where the cross term in the expansion of the quadratic factor is zero from the first equation (4). Thus at the system-level only part of the free energy is "elastic". The second term is variously termed the "frozen elastic energy" or "stored plastic work" (Maugin, 1992; Ulm and Coussy, 2003; Collins, 2005a). This additional energy is never negative, and arises from the inhomogeneity of the plastic deformation on the micro-scale. It is only zero when the plastic deformations are homogeneous on the micro-scale. Note too that this frozen energy does not depend on the system plastic strain e^P, but only on the deviation of the plastic strains in the springs from this mean value. Again the uni-modal model is an exception as it only has a single micro-plastic strain. The system free energy is hence seen to be "decoupled", and can be written as the sum of an elastic and plastic term:

$$\overline{\Psi}[\overline{e}, e_i^{mp}] = \tfrac{1}{2}\overline{K}\overline{e}^{E^2} + \tfrac{1}{2}\Sigma K_i e_i^{mr^2} = \overline{\Psi}^E[\overline{e}^E] + \overline{\Psi}^P[e_i^{mr}] \quad (6)$$

From (1) the rate of change of the free energy can be written:

$$\dot{\overline{\Psi}}[\dot{e}, e_i^{mp}] \equiv \Sigma K_i e_i^{me}\dot{e}_i^{me} = \Sigma\sigma_i(\dot{\overline{e}} - \dot{e}_i^{mp}) = \overline{\sigma}\dot{\overline{e}} - \Sigma\sigma_i\dot{e}_i^{mp} = \widetilde{W} - \widetilde{\Phi} \quad (7)$$

where $\widetilde{W} \equiv \overline{\sigma}.\dot{\overline{e}}$ is the rate of working of the applied system stress and $\widetilde{\Phi} \equiv \Sigma\sigma_i e_i^{mp}$ is the rate of dissipation on the plastic sliders. (Note we use the "tilde" notation, instead of the more usual "dot" notation for these "time-derivatives", to emphasize that these derivatives are path dependent, and there are no such functions as "the work W" and "the dissipation Φ"). Equation (7) hence states that the rate of working is equal to the sum of the rate of change of the free energy and the rate of dissipation. This, together with the inequality $\widetilde{\Phi} \geq 0$, is just the basic statement of the laws of thermodynamics for isothermal deformations.

Using (4) and (6) the rate of change of the free energy can also be written:

$$\dot{\Psi} = \dot{\overline{\Psi}}^E(e^E) + \dot{\overline{\Psi}}^P(e_i^{mr}), \text{ where}$$

$$\dot{\overline{\Psi}}^E = \overline{K}\overline{e}^E\dot{\overline{e}}^E, \text{ and } \dot{\overline{\Psi}}^P = \Sigma K_i e_i^{mr}\dot{e}_i^{mr} = \Sigma\sigma_i^r\dot{e}_i^{mr} = -\Sigma\sigma_i^r\dot{e}_i^{mp} \quad (8)$$

The applied work rate is the sum of elastic and plastic work rates, which are given by:

$$\widetilde{W}^E \equiv \overline{\sigma}\dot{\overline{e}}^E = \dot{\overline{\Psi}}^E, \text{ and } \widetilde{W}^P \equiv \overline{\sigma}\dot{\overline{e}}^P = \dot{\overline{\Psi}}^P + \widetilde{\Phi} = \Sigma(\sigma_i - \sigma_i^r)\dot{e}_i^{mp} \quad (9)$$

The plastic work rate is hence only equal to the dissipation rate when the stored plastic work is not changing. This illustrates the essential difference between plastic work and plastic dissipation; a distinction, which is frequently overlooked in the construction of geo-mechanical models as discussed in Collins, 2005a and 2005b. It is, of course, recognized that not all the springs will be deforming plastically, and these would not contribute to the sums in (8) and (9).

3. Multi-mode systems with non-linear elasticity

The total free energy of the system is now given by

$$\overline{\Psi}(\overline{e}, e_i^{mp}) = \Sigma\Psi_i(e_i^{me}) = \Sigma\Psi_i(\overline{e} - e_i^{mp}) \quad (10)$$

where Ψ_i is the free energy function for the i^{th} spring. A similar, non-linear structure of the free energy was explored by Einav (2004b). It is important to note that the use of the argument $(\overline{e} - e_i^{mp})$ in the free energies of the individual springs, reflects the fact that these energies depend just on the local elastic strain. It is quite different from assuming that the free energy of the *system* depends just on the system elastic strain. The total stress is

$$\overline{\sigma}(\dot{e}, e_i^{mp}) \equiv \Sigma\sigma_i = \Sigma\Psi_i'\ (\overline{e} - e_i^{mp}) \tag{11}$$

Because the function Ψ' is now a non-linear function of the elastic micro-strains, it is not now possible to define system elastic and plastic strains as in equation (3). However it is still possible to decompose the total strain rate into plastic and elastic parts. Differentiating (11) with respect to time gives:

$$\begin{aligned}\dot{\overline{\sigma}}(\overline{e}, e_i^{mp}) &\equiv \Sigma\dot{\sigma}_i \\ &= \Sigma(\Psi_i''\ (\overline{e} - e_i^{mp})).(\dot{\overline{e}} - \dot{e}_i^{mp}) \\ &= \dot{e}.\Sigma\Psi_i''\ (\overline{e} - e_i^{mp}) - \Sigma(\Psi_i''\ (\overline{e} - e_i^{mp}).\dot{e}_i^{mp}))\end{aligned} \tag{12}$$

We can hence define the system elastic strain rate $\dot{\overline{e}}^E$, as that strain rate which would occur if there were no instantaneous plastic deformations (ie $\dot{e}_i^{mp} = 0_i$ for all i), so:

$$\dot{\overline{\sigma}}(e, e_i^{mp}) = \dot{\overline{e}}^E.\Sigma\Psi_i''\ (e - e_i^{mp}) \text{ or } \dot{\overline{e}}^E = \dot{\overline{\sigma}}(e, e_i^{mp}) / \Sigma\Psi_i''\ (e - e_i^{mp}) \tag{13}$$

whilst the system plastic strain rate is that which pertains at constant system stress ($\dot{\overline{\sigma}} = 0$):

$$\dot{\overline{e}}^P = \Sigma(\Psi_i''\ ((\overline{e} - e_i^{mp}).\dot{e}_i^{mp})) / \Sigma\Psi_i''\ (\overline{e} - e_i^{mp}) \tag{14}$$

so that the basic decomposition of the strain *rate* is still valid at the system level:

$$\dot{\overline{e}} = \dot{\overline{e}}^E + \dot{\overline{e}}^P \tag{15}$$

However, since, both the elastic and plastic strain rates depend on the current values of the micro-strains, the elastic and plastic strains obtained by integrating these expressions are path dependent, and *cannot be used as state variables* in a continuum or thermomechanical formulation. In the linear model, $\Psi_i'' = K_i$ is a constant, and this problem does not arise.

The residual elastic strain rates (or deviation plastic micro-strains) $\dot{e}_i^{mr}$ are defined by:

$$\dot{\overline{e}}^p = \dot{e}_i^{mp} + \dot{e}_i^{mr}, \text{ where } \sum\dot{\sigma}_i^r = \sum\Psi_i''\ (\overline{e} - e_i^{mp}).\dot{e}_i^{mr} = 0 \tag{16}$$

The rate of change of the total free energy is hence:

$$\begin{aligned}\dot{\overline{\Psi}}(\overline{e}, e_i^{mp}) &= \sum\Psi'(\overline{e} - e_i^{mp}).(\dot{\overline{e}} - \dot{e}_i^{mp}) = \sum\Psi'(\overline{e} - e_i^{mp}).(\dot{\overline{e}}^E + \dot{e}_i^{mr}) \\ &= \overline{e}^E.\sum\Psi'(\overline{e} - e_i^{mp}) + \sum\Psi'(\overline{e} - e_i^{mp}).\dot{e}_i^{mr} = \overline{\sigma}.\dot{\overline{e}}^E + \sum\sigma_i e_i^{mr}\end{aligned} \tag{17}$$

using (15). We note that although the rate of change of the free energy can be split into elastic and plastic parts, the time integrals of these two functions are path independent so that the functions $\overline{\Psi}^E(e^E)$ and $\overline{\Psi}^P(e_i^{mr})$ are *not state functions*. However, from the basic thermomechanical energy balance equation, we also know that:

$$\dot{\overline{\Psi}}(\bar{e}, e_i^{mp}) = \bar{\sigma}\dot{\bar{e}} - \tilde{\Phi} = \bar{\sigma}\dot{\bar{e}}^{E} + \bar{\sigma}\dot{\bar{e}}^{p} - \sum \sigma_i \dot{e}_i^{mp} \tag{18}$$

Comparison of these last two equations shows that the rate of plastic work is given by:

$$\bar{\sigma}\dot{\bar{e}}^{p} = \sum \sigma_i \dot{e}_i^{mr} + \sum \sigma_i \dot{e}_i^{mp} \tag{19}$$

The second term is the rate of dissipation, the first is the rate of change of frozen elastic energy or stored plastic work, arising from the inhomogeneities of the plastic strain rates on the micro-scale.

There is hence no intrinsic difference between the *rate formulations* for linear and non-linear micro-elastic materials. The differences occur in the integrated formulations. Specifically, for nonlinear models, the elastic and plastic strains are not state variables, and the elastic and plastic parts of the free energy are not state functions.

In the next section these arguments are recast, using the standard notation of homogenization theory.

4. Homogenization analyses.

The mean value of a quantity averaged over a representative volume element (RVE) will be denoted by:

$$\langle * \rangle \equiv \frac{1}{\Delta V}\int_{\Delta V} * \; dV \tag{20}$$

Provided a micro-level stress field is statically admissible the continuum stress is equal to the average of the micro stress field. The corresponding results hold for micro-level strain or strain rate fields, provided they are kinematically admissible. As above, we invoke the Voigt homogenization procedure and assume the strain is uniform in the RVE.

The micro elastic behaviour is characterized by a free energy function $\Psi(\bar{e} - e^{mp})$, so that the local stress in the RVE is $\sigma = \Psi'(\bar{e} - e^{mp})$. The continuum stress and stress rate are hence:

$$\bar{\sigma} = \langle \Psi'(\bar{e} - e^{mp}) \rangle, \text{ and } \dot{\bar{\sigma}} = \langle \Psi''(e - e^{mp}) \rangle : \dot{e} - \langle \Psi''(e - e^{mp}) : \dot{e}^{mp} \rangle \tag{21}$$

where Ψ' and Ψ'' are the second and fourth order derivative tensors, respectively.

The *continuum* elastic strain rate, defined as the strain rate pertaining when there is no micro-plastic strain rates, and the *continuum* plastic strain rate, defined as the strain rate occurring when the *continuum* stress rate is zero, are hence given by:

$$\dot{\bar{e}}^{E} = \langle \Psi''(\bar{e} - e^{mp}) \rangle^{-1} : \dot{\bar{\sigma}}, \text{ and } \dot{\bar{e}}^{P} = \langle \Psi''(\bar{e} - e^{mp}) \rangle^{-1} : \langle \Psi''(\bar{e} - e^{mp}) : \dot{e}^{mp} \rangle \tag{22}$$

respectively. It follows that $\dot{e} = \dot{\bar{e}}^{E} + \dot{\bar{e}}^{P}$, but since these two rates depend on the current state as defined by e and e^{mp}, they cannot be integrated to give path independent, elastic and plastic strains. These results are the continuum analogue of equations (13) - (15). The continuum plastic strain rate is not the simple average of the plastic micro-strain rates, since the latter are not kinematically admissible.

The rate of change of the continuum free energy is

$$\dot{\overline{\Psi}} = <\dot{\Psi}(\overline{e}-e^{mp})> \; = <\Psi'(\overline{e}-e^{mp}):(\dot{\overline{e}}-\dot{e}^{mp})> = \overline{\sigma}:\dot{\overline{e}} - <\sigma:\dot{e}^{mp}> \qquad (23)$$

so that it is equal to the stress power minus the rate of energy dissipated by the micro-strains, as required by the laws of thermodynamics. The stress power is the sum of the rates of elastic and plastic working. The elastic work rate is equal to the rate of change of the elastic "part" of the free energy:

$$\dot{\tilde{\overline{\Psi}}}^{E} = \overline{\sigma}:\dot{\overline{e}}^{E} \; = \overline{\sigma}:<\Psi''(\overline{e}-e^{mp})>^{-1}:\dot{\overline{\sigma}} \qquad (24)$$

However, since this derivative depends on the current micro-strains, its integral is path dependent and $\overline{\Psi}^{E}$ is *not* a state function: nor is $\overline{\Psi}^{P}$ - the plastic part of the free energy function.

From (23) and (24) the rate of change of the "plastic part" of the free energy is the difference between the rate of plastic work and the rate of dissipation, and can be written:

$$\dot{\tilde{\overline{\Psi}}}^{P} = <\overline{\sigma}:(\dot{\overline{e}}^{P} - \dot{e}^{mp})>$$

$$= \; \overline{\sigma}:<\Psi''(\overline{e}-e^{mp})>^{-1}:\{<\Psi''(\overline{e}-e^{mp}):\dot{e}^{mp}> - <\Psi''(\overline{e}-e^{mp})>:<\dot{e}^{mp}>\} \qquad (25)$$

5. Conclusions

In order to understand the true physical significance of the continuum concepts of elastic and plastic strain, it is necessary to define them as certain averages over the micro strains occurring within a typical "continuum element" or "representative volume element". This is particularly important for soils and sands, where the micro-level stress and strain distributions are highly inhomogeneous, as evidenced by the formation of weak and strong force networks. By appealing to the "system analogy" of sets of springs and dashpots, and by using standard homogenization theory, it has been demonstrated that the assumption that the continuum elastic and plastic strains may be used as state variables is only justified when the micro-elastic behaviour is linear. In this case the free energy can also be written as the sum of an "elastic" and "plastic" term. When the micro-elastic behaviour is non-linear however, it is necessary to start with a rate formulation, as the continuum elastic and plastic strains are path dependent, and are not uniquely defined by the current state of the soil continuum.

References

Collins, I. F. (2005a): The concept of "stored plastic work" or "frozen elastic energy" in soil mechanics. Geotechnique (in press).

Collins, I. F. (2005b): Elastic/plastic models for soils and sands. Int. J. mech. Sciences (in press)

Collins, I.F. and Houlsby, G.T. (1997): Application of thermomechanical principles to the modelling of geotechnical materials. Proc. Roy. Soc. Lond. *A* **453 ,**1975-2001.

Einav, I. (2004a): Thermomechanical relations between basic stress-space and strain-space models. Geotechnique, **54** (5)., 315-318.

Einav, I. (2004b): A second look at strain space plasticity and latest applications. Developments in Mechanics of Structures and Materials, Perth, (1), 225 - 231.

Iwan, W. D. (1966): A distributed element model for hysteresis and its steady state response. J. Appl. Mech. Trans ASME, **33**, 893-900.

Iwan, W. D. (1967): On a class of models for the yielding behaviour of continuous and composite systems. J. Appl. Mech. Trans ASME, **34**, 612-617.

Johnson, K.L. (1985): Contact Mechanics, Cambridge University Press.

Masing, G. (1926): Eigenspannungen und verfestigung beim messing. Proc 2nd Int. Congr. Appl. Mech.

Maugin, G.A. (1992): The thermomechanics of plasticity and fracture, Cambridge University Press.

Mroz, Z. (1973): Mathematical models of inelastic material behaviour, University of Waterloo Press.

Oda, M and Iwashita, K. (1999): An Introduction to Granular Materials, Balkema, Rotterdam.

Ulm, F-J, and Coussy, O. (2003): Mechanics and durability of solids, Vol 1. New Jersey, Prentice Hall